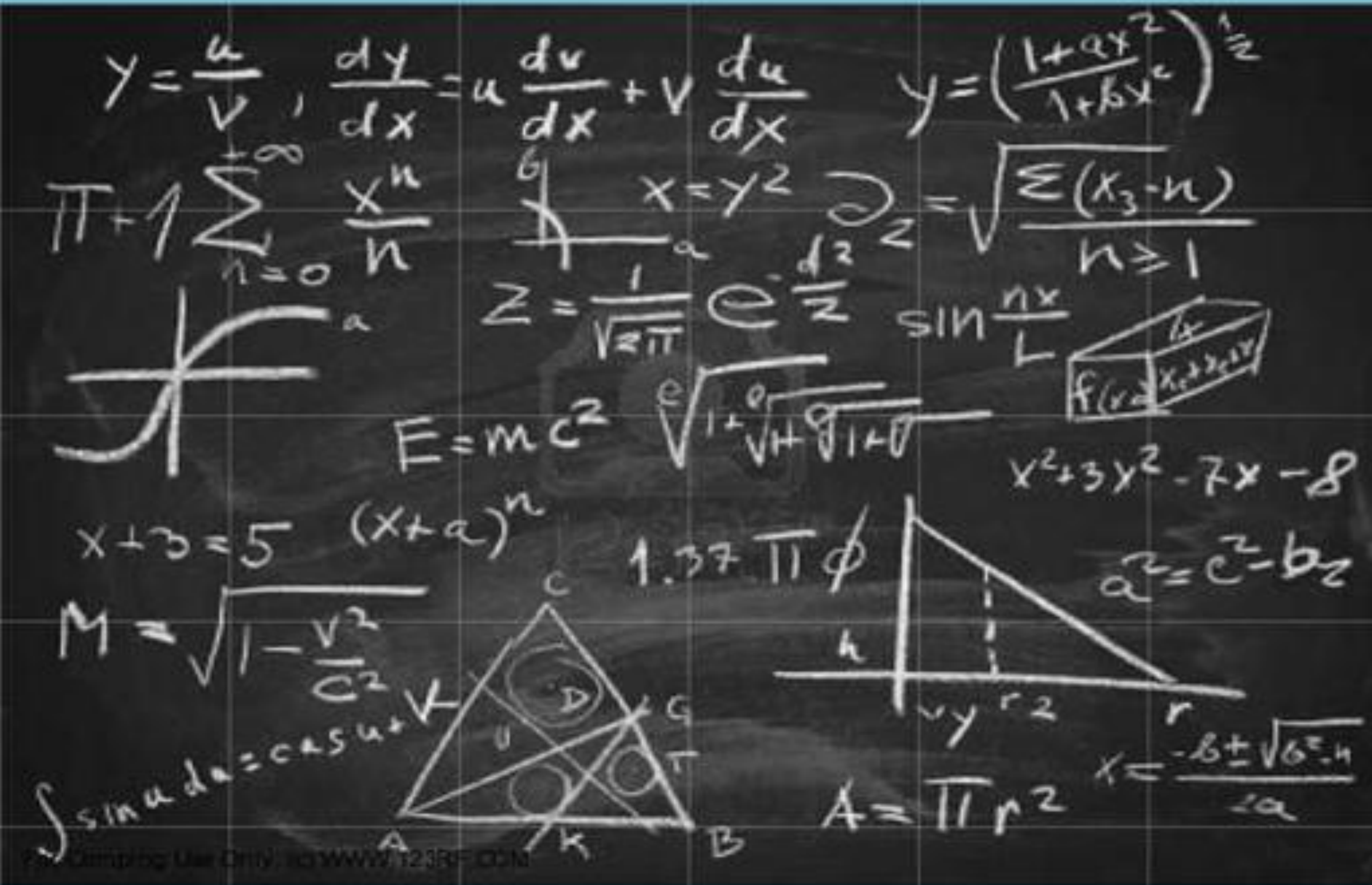
Development of e-Courses for B.Sc.(Agriculture) Degree Program



MATHEMATICS

Mathematics

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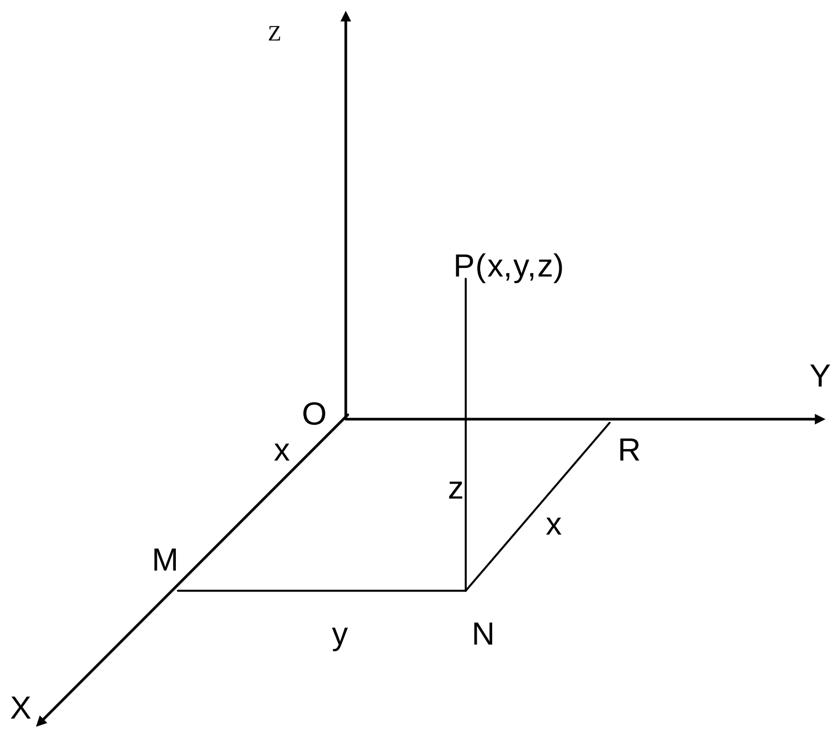
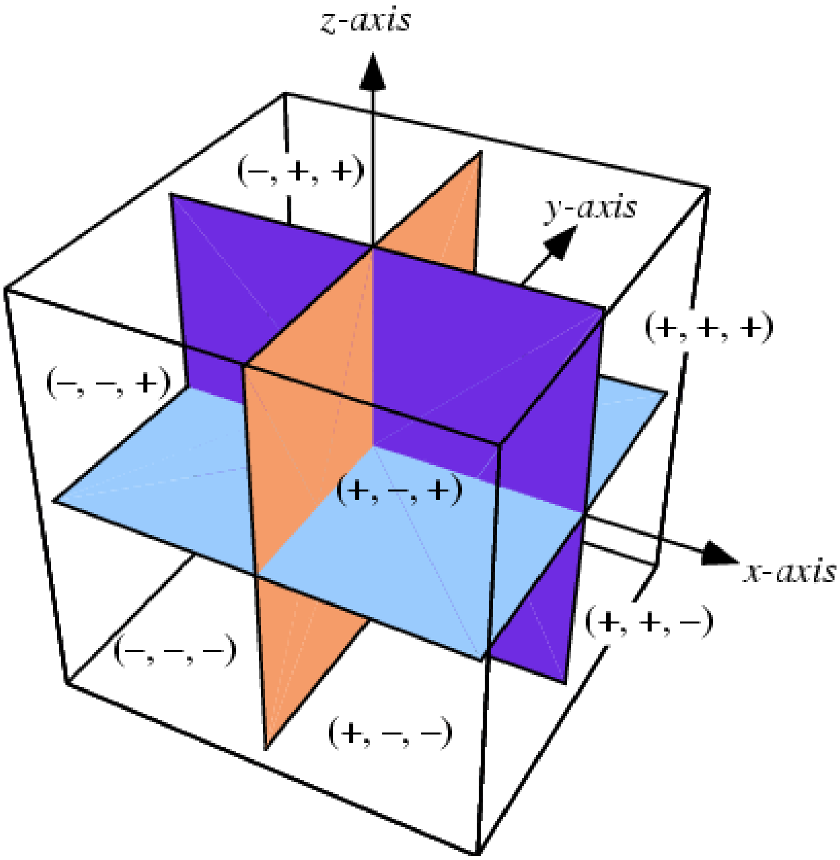
INDEX

Three dimensional Analytical geometry

Let OX ,OY & OZ be mutually perpendicular straight lines meeting at a point O. The extension of these lines OX^1, OY^1 and OZ^1 divide the space at O into octants(eight). Here mutually perpendicular lines are called X, Y and Z co-ordinates axes and O is the origin. The point P (x, y, z) lies in space where x, y and z are called x, y and z coordinates respectively.



where NR = x coordinate, MN = y coordinate and PN = z coordinate



Distance between two points

The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

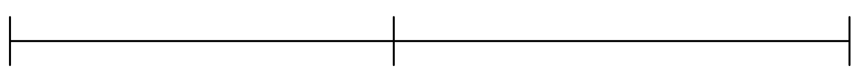
dist AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

In particular the distance between the origin O (0,0,0) and a point P(x,y,z) is

OP = $\sqrt{x^2 + y^2 + z^2}$

The internal and External section

Suppose $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points in three dimensions.



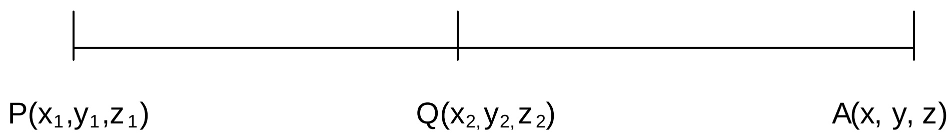
$P(x_1, y_1, z_1)$ A(x, y, z) $Q(x_2, y_2, z_2)$

The point A(x, y, z) that divides distance PQ internally in the ratio $m_1 : m_2$ is given by

$$A = \left[\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}, \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}\right]$$

Similarly

$P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points in three dimensions.



The point A(x, y, z) that divides distance PQ externally in the ratio $m_1 : m_2$ is given by

$$A = \left[\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}, \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}\right]$$

If A(x, y, z) is the midpoint then the ratio is 1:1

$$A = \left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right]$$

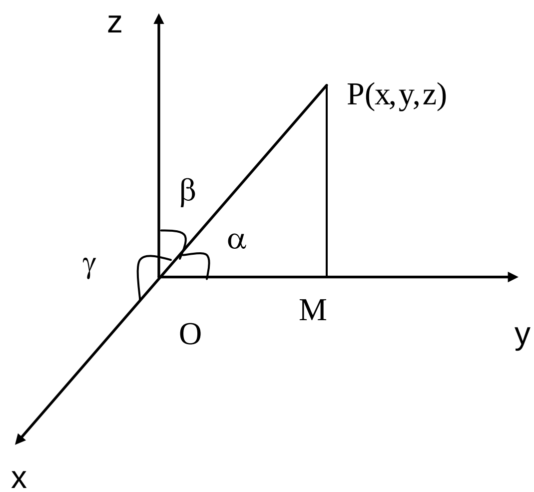
Problem

Find the distance between the points P(1,2-1) & Q(3,2,1)

PQ= $\sqrt{(3-1)^2 + (2-2)^2 + (1+1)^2} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

Direction Cosines

Let P(x, y, z) be any point and OP = r. Let α,β,γ be the angle made by line OP with OX, OY & OZ. Then α,β,γ are called the direction angles of the line OP. cos α, cos β, cos γ are called the direction cosines (or dc's) of the line OP and are denoted by the symbols l, m ,n.



Result

By projecting OP on OY, PM is perpendicular to y axis and the $\angle POM = \beta$ also OM = y

$$\therefore \cos\beta = \frac{y}{r}$$

Similarly, $$\cos\alpha = \frac{x}{r}$$

$$\cos\gamma = \frac{z}{r}$$

(i.e) $l = \frac{x}{r}$, $m = \frac{y}{r}$, $n = \frac{z}{r}$

$$\therefore l^2 + m^2 + n^2 = \frac{x^2+y^2+z^2}{r^2}$$

($\because r = \sqrt{x^2+y^2+z^2} \Rightarrow$ Distance from the origin)

$$\therefore l^2 + m^2 + n^2 = \frac{x^2+y^2+z^2}{x^2+y^2+z^2} = 1$$

$$l^2 + m^2 + n^2 = 1$$

(or) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

<u>**Note**</u> **:-**

The direction cosines of the x axis are (1,0,0)

The direction cosines of the y axis are (0,1,0)

The direction cosines of the z axis are (0,0,1)

Direction ratios

Any quantities, which are proportional to the direction cosines of a line, are called direction ratios of that line. Direction ratios are denoted by a, b, c.

If l, m, n are direction cosines an a, b, c are direction ratios then

$a \propto l,\ b \propto m,\ c \propto n$

(ie) $a = kl,\ b = km,\ c = kn$

(ie) $\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = k$ (Constant)

(or) $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{1}{k}$ (Constant)

To find direction cosines if direction ratios are given

If a, b, c are the direction ratios then direction cosines are

$$\left.\begin{aligned} \frac{l}{a} = \frac{1}{k} \Rightarrow \quad & l = \frac{a}{k} \\ \textit{similarly} \quad & m = \frac{b}{k} \\ & n = \frac{c}{k} \end{aligned}\right\} \qquad (1)$$

$$l^2+m^2+n^2 = \frac{1}{k^2}(a^2+b^2+c^2)$$

(ie) $$1 = \frac{1}{k^2}(a^2+b^2+c^2)$$

$$\Rightarrow k^2 = a^2+b^2+c^2$$

Taking square root on both sides

$$K = \sqrt{a^2+b^2+c^2}$$

$$\therefore\ l = \frac{a}{\sqrt{a^2+b^2+c^2}},\quad m = \frac{b}{\sqrt{a^2+b^2+c^2}},\quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

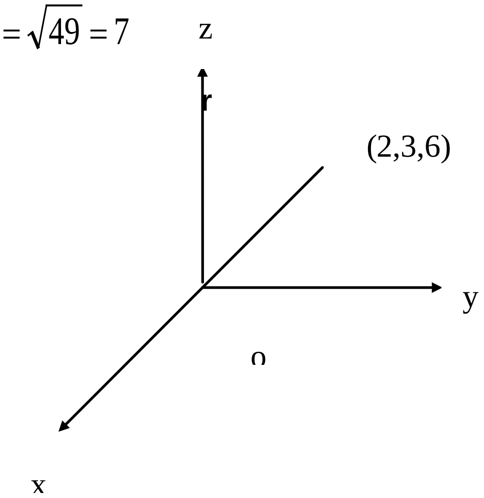
Problem

1. Find the direction cosines of the line joining the point (2,3,6) & the origin.

Solution

By the distance formula

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$



Direction Cosines are

$$l = \cos \alpha = \frac{x}{r} = \frac{2}{7}$$

$$m = \cos \beta = \frac{y}{r} = \frac{3}{7}$$

$$n = \cos \gamma = \frac{z}{r} = \frac{6}{7}$$

2. Direction ratios of a line are 3,4,12. Find direction cosines

Solution

Direction ratios are 3,4,12

(ie) a = 3, b = 4, c = 12

Direction cosines are

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{3}{\sqrt{3^2 + 4^2 + 12^2}} = \frac{3}{\sqrt{169}} = \frac{3}{13}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{4}{\sqrt{3^2 + 4^2 + 12^2}} = \frac{4}{\sqrt{169}} = \frac{4}{13}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{12}{\sqrt{3^2 + 4^2 + 12^2}} = \frac{12}{\sqrt{169}} = \frac{12}{13}$$

Note

1) The direction ratios of the line joining the two points A(x_1, y_1, z_1) & B (x_2, y_2, z_2) are ($x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$)

2) The direction cosines of the line joining two points A (x_1, y_1, z_1) & B (x_2, y_2, z_2) are $\frac{x_2 - x_1}{r}, \frac{y_2 - y_1}{r}, \frac{z_2 - z_1}{r}$

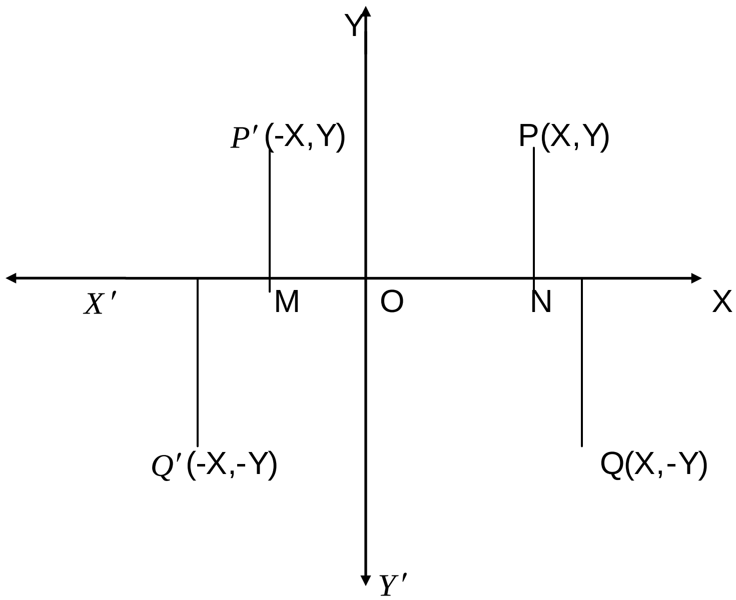
r = distance between AB.

Analytical geometry

Introduction

The branch of mathematics where algebraic methods are employed for solving problems in geometry is known as analytical geometry. It is sometimes called Cartesian Geometry.

Let X'OX and Y'OY be two perpendicular straight lines intersecting at the point O. The fixed point O is called origin. The horizontal line X'OX is known as X –axis and the vertical line Y'OY be Y-axis. These two axes divide the entire plane into four parts known as Quadrants.



All the values right of the origin along the X-axis are positive and all the values left of the origin along the X- axis are negative. Similarly all the values above the origin along Y – axis are positive and below the origin are negative.

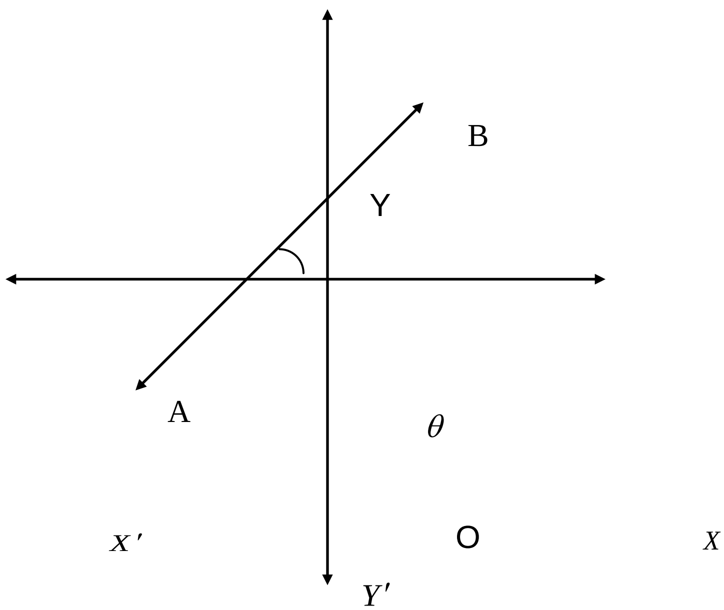
Let P be any point in the plane. Draw PN perpendicular to X –axis. ON and PN are called X and Y co-ordinates of P respectively and is written as P (X,Y). In particular the origin O has co-ordinates (0,0) and any point on the X-axis has its Y co-ordinate as zero and any point on the Y-axis has its X-co-ordinates as zero.

Straight lines

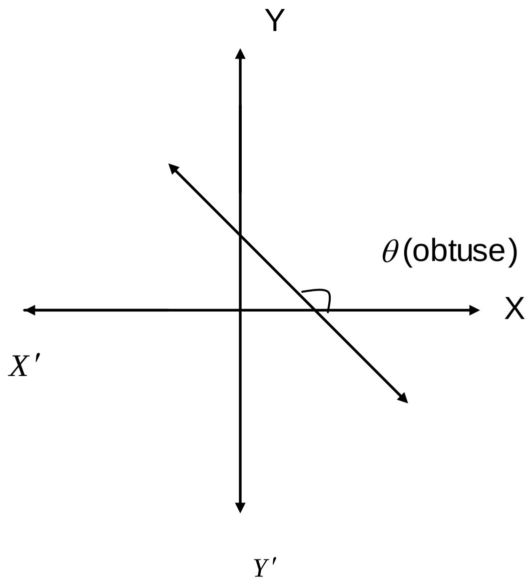
A straight line is the minimum distance between any two points.

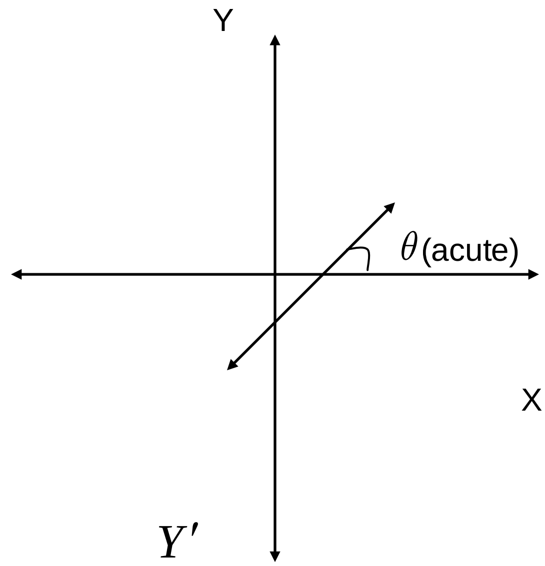
Slope

The slope of the line is the tangent of the angle made by the line with positive direction of X – axis measured in the anticlockwise direction.



let the line AB makes an angle θ with the positive direction of X-axis as in the figure. The angle θ is called the angle of inclination and tan θ is slope of the line or gradient of the line. The slope of the line is denoted by m. i.e., slope = m = tan θ





Slope = m = tan θ Slope = m= tan θ Slope is negative Slope is positive

Note

(i) The slope of any line parallel to X axis is zero.

(ii) Slope of any line parallel to Y axis is infinity

(iii) The slope of the line joining two points (x_1, y_1) and (x_2, y_2) is

$$\text{Slope} = m = \tan\theta = \frac{y_1 - y_2}{x_1 - x_2}$$

(iv) When two or more lines are parallel then their slopes are equal

(v) When two lines are perpendicular then the product of their slopes is –1

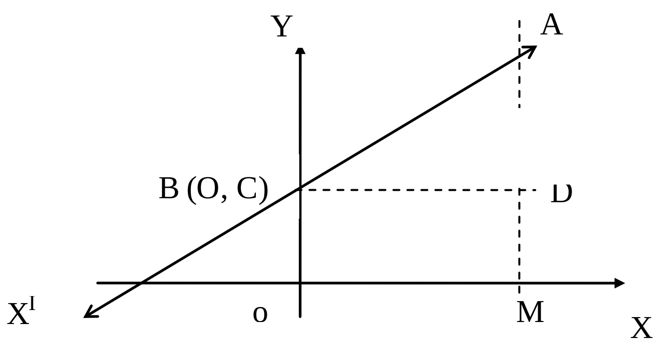
i.e., $m_1 m_2 = -1$

Equation of a straight line

There are several forms of a straight line. They are,

1. Slope – intercept form

Let the given line meet y-axis at B (o, c). We call OB as Y – intercept. Let A be any point on the given line. Draw AM perpendicular to OX and BD $\perp$ AM. Let this line make an angle θ with X axis. Then the slope,



$$m = \tan\theta = \frac{AD}{BD} = \frac{AM - DM}{OM} = \frac{AM - OB}{OM} = \frac{y-c}{x}$$

ie. $m = \dfrac{y-c}{x}$

$mx = y - c$

$y = mx + c$

Hence, the equation of a line with slope 'm' and y – intercept 'c' is given by

$y = mx + c$

Note:

(i) Any line passing through the origin does not cut y – axis (c = 0) i.e., y – intercept is zero. Therefore its equation is y = mx

(ii) Any line which is parallel to x – axis has slope equal to zero. Therefore its equation is y = c (Because m = 0)

(iii) Any line perpendicular to x-axis, ie which is parallel to y-axis at a distance of K units from the origin is given by x = k.

Example 1: Find the equation of a straight line whose

(i) Slope is four and y intercept is –3

(ii) Inclination is 30^0 and y intercept is 5

Solution: (i) Slope (m) = 4

Y intercept (c) = -3

Equation of a line is y = mx + c

Y = 4x – 3

Equation of a line is 4x – y – 3 = 0

(ii) $\theta = 30^0$, y intercept = 5

Slope = tan θ

m = tan 30^0 = $\frac{1}{\sqrt{3}}$

Equation of a line is y = mx + c

Y = $\frac{1}{\sqrt{3}}$x + 5

$\sqrt{3}$y = x + 5$\sqrt{3}$

Equation of a line is x - $\sqrt{3}$y +5 $\sqrt{3}$ = 0

Example 2: Calculate the slope and y intercept of the line 2x – 3y + 1= 0

Solution: 2x – 3y + 1 = 0

3y = 2x + 1

y = $\frac{2x}{3}$ + $\frac{1}{3}$

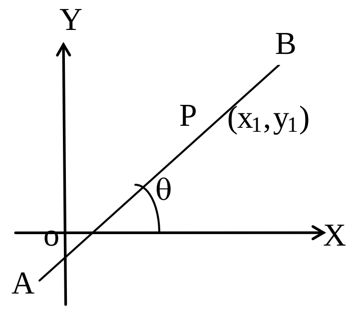
Comparing with y = mx + c, we get

m = $\frac{2}{3}$, c = $\frac{1}{3}$

Slope = $\frac{2}{3}$; y intercept = $\frac{1}{3}$

2. Slope – one point form

Let the line AB make an angle θ with x- axis as shown in the figure and pass through the point P (x_1, y_1). If (x, y) represents a point other than the point (x_1, y_1), then m = $\frac{y-y_1}{x-x_1}$ where m is the slope of the line or $y - y_1 = m(x - x_1)$.



Hence the equation of a line passing through a point (x_1, y_1) and having slope ‘m’ is

$y - y_1 = m\,(x - x_1)$.

Example: Find the equation of a straight line passing through (-4,5) and having slope $-\frac{2}{3}$

Solution: Slope = $-\frac{2}{3}$

Point (-4,5)

Equation of the line is $(y-y_1) = m(x-x_1)$

$$y - 5 = -\frac{2}{3}(x+4)$$

$$3y - 15 = -2x - 8$$

$\therefore$ Equation of a line is $2x + 3y - 7 = 0$

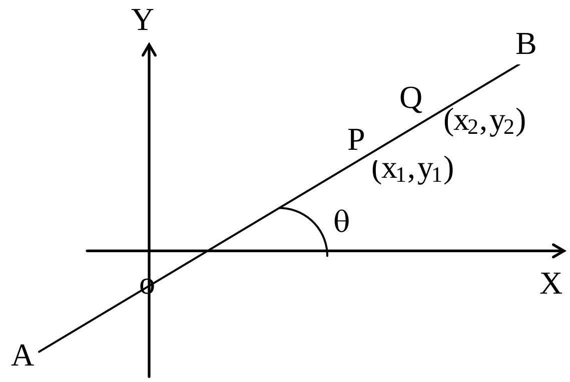
3. Two points form

Let P (x_1, y_1) and Q (x_2, y_2) be any two points on the given line AB. We know, the slope, $m = \frac{y_2 - y_1}{x_2 - x_1}$.

We have the slope-point form of a line as

$$y - y_1 = m\,(x - x_1).$$

Substituting the value of m in the above equation we get,



$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

ie $$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Hence, the equation of a line passing through two points is given by

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Example: Find the equation of the straight line passing through the points (3,6) and (-2,5).

Solution : Equation of the line is $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

$$\frac{y-6}{5-6} = \frac{x-3}{-2-3}$$

$$\frac{y-6}{-1} = \frac{x-3}{-5}$$

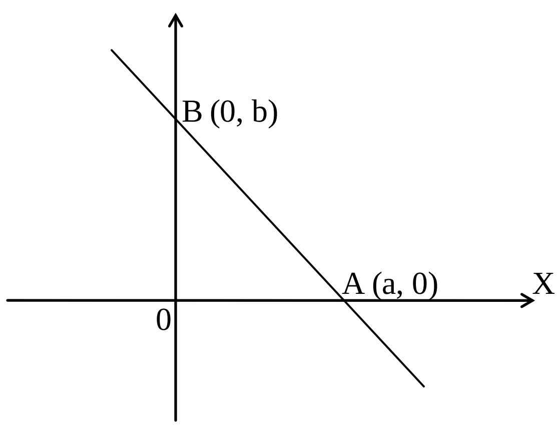
$$5y - 30 = x - 3$$

$$x - 5y - 3 + 30 = 0$$

Equation of the line is $x - 5y + 27 = 0$

4. Intercept form

Let AB represent the given line which intersects X – axis at A (a, 0) and Y- axis at B (0, b). We call OA and OB respectively as x and y intercepts of the line.



The two points form of the equation is given by $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

Substituting (a, 0) for (x_1, y_1) and (0, b) for (x_2, y_2), we get the equation as

$$\frac{y-0}{b-0} = \frac{x-a}{0-a}$$

ie $\frac{y}{b} = \frac{x-a}{-a}$

$$\frac{y}{b} = \frac{x}{-a} - \frac{a}{-a}$$

Thus, $\frac{y}{b} = \frac{-x}{a} + 1$

$$\therefore \quad \frac{x}{a} + \frac{y}{b} = 1$$

Hence, the equation of a line having x-intercept ‘a’ and y-intercept ‘b’ is given by

$$\frac{x}{a} + \frac{y}{b} = 1$$

Example: Find the intercepts cut off by the line 2x – 3y + 5 = 0 on the axes.

Solution

x – intercept: put y = 0

$\therefore$ 2x + 5 = 0

x = $\frac{-5}{2}$ This is the x – intercept

y – intercept: Put x = 0

-3y + 5 = 0

$\therefore$ y = $\frac{5}{3}$ This is the y – intercept

Example : Give the mathematical equation of the supply function of a commodity such that the quantity supplied is zero when the price is Rs.5 or below and it increase continuously at the constant rate of 10 units for each one rupee rise in price above Rs.5.

Solution

Point B is (10,6)

Point C is (20,7)

Equation of straight line joining two points is $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

$$\frac{y-6}{7-6} = \frac{x-10}{20-10}$$

$$\frac{y-6}{1} = \frac{x-10}{10}$$

10 (y-6) = x –10

10y-60 = x-10

x -10y - 50=0

Note

The four equations we have obtained are all first degree equations in x and y. On the other hand it can be shown that the general first degree equation in x and y always represents a straight line. Hence we can take general equation of a straight line as ax + by + c = 0 with at least one of a or b different from c. Further, this gives by = - ax – c

i.e. $y = \frac{-ax}{b} - \frac{c}{b}$

Now, comparing this with the equation y = mx + c, we get

slope = m = $\frac{-a}{b} = -\left[\frac{coefficient\ of\ x}{coefficient\ of\ y}\right]$

BINOMIAL THEOREM

A Binomial is an algebraic expression of two terms which are connected by the operation '+' (or) '-'

For example, *x+siny, 3x²+2x, cosx+sin x* etc… are binomials.

Binomial Theorem for positive integer:

If n is a positive integer then

$(x+a)^n = nC_0 x^n a^0 + nC_1 x^{n-1} a^1 + \ldots + nC_r x^{n-r} a^r + \ldots + nC_{n-1} x^1 a^{n-1} + nC_n x^0 a^n$	----(1)

Some Expansions

a) If we put a = -a in the place of a in

$$(x-a)^n = nC_0 x^n (-a)^0 + nC_1 x^{n-1} (-a)^1 + \ldots + nC_r x^{n-r} (-a)^r + \ldots + nC_{n-1} x^1 (-a)^{n-1} + nC_n x^0 (-a)^n$$

$$(x-a)^n = nC_0 x^n a^0 - nC_1 x^{n-1} a^1 + \ldots + (-1)^r nC_r x^{n-r} a^r + \ldots + (-1)^r nC_{n-1} x^1 a^{n-1} + \ldots + (-1)^r nC_n x^0 a^n$$

b) Put x =1 and a = x in (1)

$$(1+x)^n = 1 + nC_1 x + nC_2 x^2 + \ldots + nC_r x^r + \ldots + nC_n x^n$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 \ldots\ldots + x^n \quad \text{----------(2)}$$

c) Put x = 1 and a = -x in (1)

$$(1-x)^n = 1 - nC_1 x + nC_2 x^2 - \ldots + (-1)^r nC_r x^r + \ldots + (-1)^n nC_n x^n$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!} x^2 - \frac{n(n-1)(n-2)}{3!} x^3 \ldots\ldots + (-1)^n x^n \quad \text{-----------(3)}$$

(d) Replacing n by – n in equation (2)

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 \ldots\ldots + (-1)^n x^n \quad \text{---------(4)}$$

e) Replacing n by – n in equation (3)

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 \ldots\ldots + x^n \quad \text{-----------(5)}$$

Special Cases

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 \ldots\ldots$
2. $(1-x)^{-1} = 1 + x + x^2 + x^3 \ldots\ldots$
3. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 \ldots\ldots$
4. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 \ldots\ldots$

Note:

1. There are *n+1* terms in the expansion of $(x+a)^n$.

2. In the expansion the general term is $nC_r\, x^{n-r} a^r$. Since this is the $(r+1)^{th}$ term, it is denoted by T_{r+1} i.e. $T_{r+1} = nC_r\, x^{n-r} a^r$.

3. $nC_0, nC_1, nC_2, ...nC_r, ...nC_n$ are called binomial coefficients.

4. From the relation $nC_r = nC_{n-r}$, we see that the coefficients of terms equidistant from the beginning and the end are equal.

Note: The number of terms in the expansion of $(x+a)^n$ depends upon the index *n*. the index is either even (or) odd. Then the middle term is

Case(i): *n* is even

The number of terms in the expansion is *(n+1)*, which is odd.

Therefore, there is only one middle term and is given by $T_{\frac{n}{2}+1}$

Case(ii) : *n* is odd

The number of terms in the expansion is *(n+1),* which is even.

Therefore, there are two middle terms and they are given by $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$

Examples

1. Expand (i) $\left(2x^2 + \frac{1}{x}\right)^6$

2. Find 11^7.

Solution:

$11^7 = (1+10)^7$

$$= 7C_0(1)^7(10)^0 + 7C_1(1)^6(10)^1 + 7C_2(1)^5(10)^2 + 7C_3(1)^4(10)^3 + 7C_4(1)^3(10)^4 + 7C_5(1)^2(10)^5 + 7C_6(1)^1(10)^6 + 7C_7(1)^0(10)^7$$

$$= 1 + 70 + \frac{7\times6}{1\times2}10^2 + \frac{7\times6\times5}{1\times2\times3}10^3 + \frac{7\times6\times5\times4}{1\times2\times3\times4}10^4 + \frac{7\times6\times5\times4\times3}{1\times2\times3\times4\times5}10^5 + \frac{7\times6\times5\times4\times3\times2}{1\times2\times3\times4\times5\times6}10^6 + \frac{7\times6\times5\times4\times3\times2\times1}{1\times2\times3\times4\times5\times6\times7}10^7$$

= 1+ 70 + 2100 +35000 + 350000 + 2100000 + 7000000 + 10000000

= 19487171

2. Find the coefficient of x^5 in the expansion of $\left(x+\frac{1}{x^3}\right)^{17}$

Solution

In the expansion of $\left(x+\frac{1}{x^3}\right)^{17}$, the general term is

$$T_{r+1} = 17C_r\, x^{17-r}\left(\frac{1}{x^3}\right)^r$$

$$= 17C_r\, x^{17-4r}$$

Let T_{r+1} be the term containing x^5

then, 17-4r = 5 $\Rightarrow$ r = 3

$$\therefore \quad T_{r+1} = T_{3+1}$$

$$= 17C_r\, x^{17-4(3)} = 680\, x^5$$

$\therefore$ coefficient of x^5 = 680.

3. Find the constant term in the expansion of $\left(\sqrt{x}-\frac{2}{x^2}\right)^{10}$

Solution

In the expansion of $\left(\sqrt{x}-\frac{2}{x^2}\right)^{10}$, the general term is

$$T_{r+1} = 10C_r\left(\sqrt{x}\right)^{10-r}\left(\frac{-2}{x^2}\right)^r$$

$$= 10C_r\, x^{\frac{10-r}{2}}\frac{(-2)^r}{x^{2r}} = 10C_r\,(-2)^r\, x^{\frac{10-r}{2}-2r}$$

$$= 10C_r\,(-2)^r\, x^{\frac{10-5r}{2}}$$

Let T_{r+1} be the Constant term then,

$$\frac{10-5r}{2} = 0 \Rightarrow r = 2$$

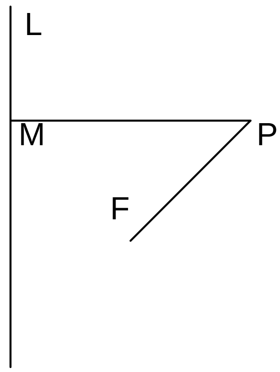
$\therefore$ The constant term $= 10C_2\,(-2)^2\, x^{\frac{10-5(2)}{2}}$

$$= \frac{10\times 9}{1\times 2}\times 4\times x^0 = 180$$

Conics

Definition

A conic is defined as the locus of a point, which moves such that its distance from a fixed line to its distance from a fixed point is always constant. The fixed point is called the focus of the **conic**. The fixed line is called the **directrix** of the conic. The constant ratio is the **eccentricity** of the conic.



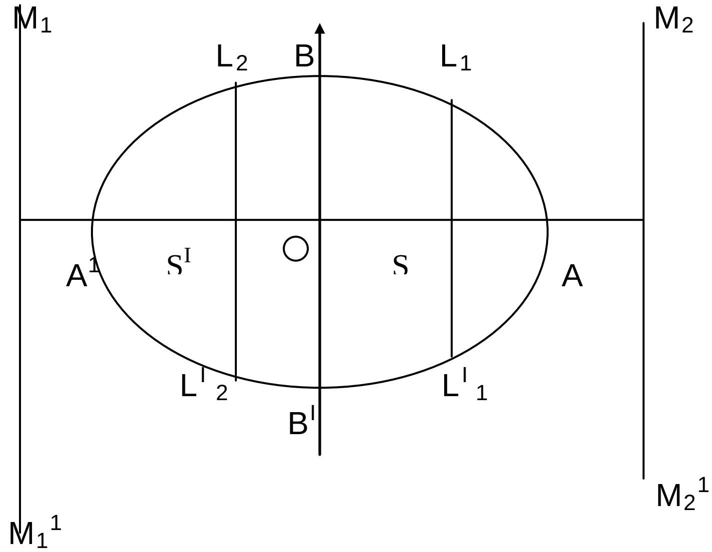
L is the fixed line – Directrix of the conic.

F is the fixed point – Focus of the conic.

$\frac{FP}{PM} =$ constant ratio is called the eccentricity = 'e'

Classification of conics with respect to eccentricity

1. If e < 1, then the conic is an Ellipse



1) The ***standard equation*** of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

2) The line segment AA^1 is the ***major axis*** of the ellipse, $AA^1 = 2a$

3) The equation of the major axis is Y = 0

4) The line segment BB^1 is the ***minor axis*** of the ellipse, BB1 = 2b

5) The equation of the minor axis is X = 0

6) The length of the major axis is always ***greater than*** the minor axis.

7) The point O is the intersection of major and minor axis.

8) The co-ordinates of O are (0,0)

9) The ***foci*** of the ellipse are S(ae,0)and S^l(-ae,0)

10) The vertical lines passing through the focus are known as ***Latusrectum***

11) The length of the Latusrectum is $\frac{2b^2}{a}$

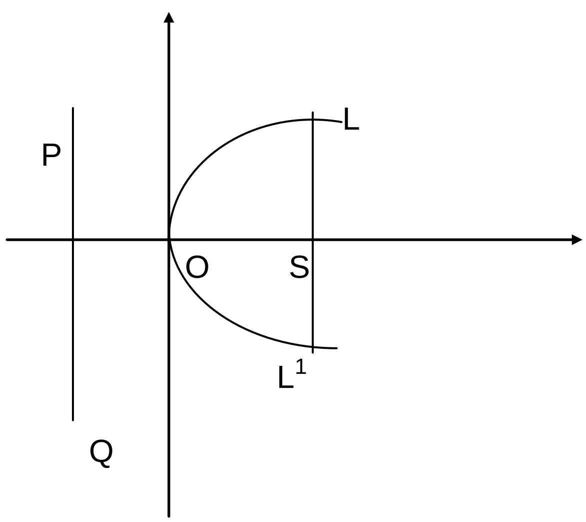
12) The points A (a,0) and A^1(-a,0)

13) The ***eccentricity*** of the ellipse is e = $\sqrt{1-\frac{b^2}{a^2}}$

14) The vertical lines $M_1M_1^1$ and $M_2M_2^1$ are known as the ***directrix*** of the ellipse and their respective

equations are $x = \frac{a}{e}$ and $x = \frac{-a}{e}$

2. If e = 1, then the conic is a **Parabola.**



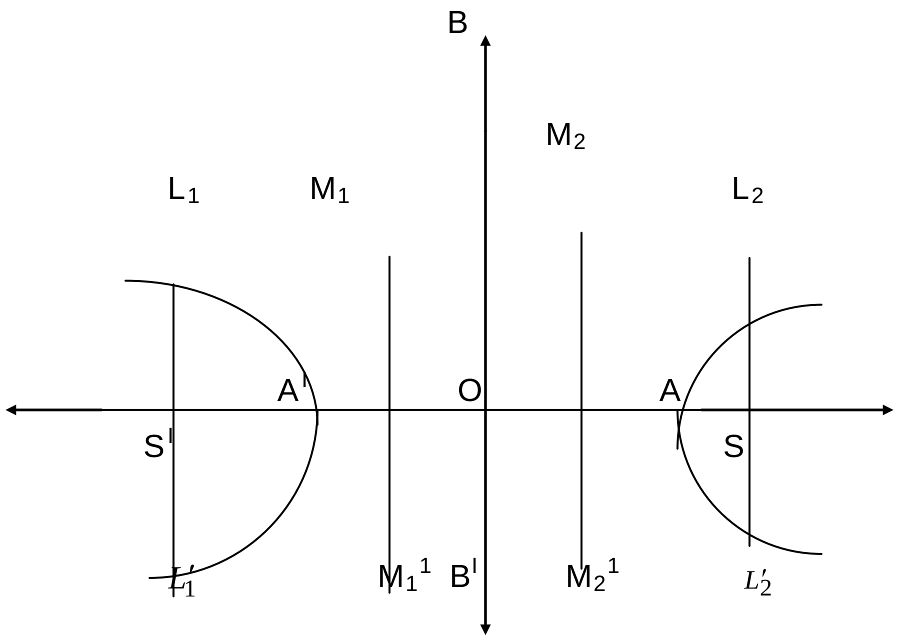
1) The ***Standard equation*** of the parabola is $y^2 = 4ax$.
2) The horizontal line is the ***axis of the parabola***.
3) The equation of the axis of the parabola is Y = 0
4) The parabola $y^2 = 4ax$ is ***symmetric*** about the axis of the parabola.
5) The ***vertex*** of the parabola is O (0,0)
6) The line PQ is called the ***directrix*** of the parabola.

7) The equation of the directrix is x = -a

8) The ***Focus*** of the parabola is S(a,0).

9) The vertical line passing through S is the ***latus rectum***. LL^1 is the Latus rectum and its length $LL^1 = 4a$

3. If e > 1 ,then the conic is ***Hyperbola.***



1) The **standard equation** of an hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

2) The line segment AA^1 is the **Transverse axis** of the hyperbola ,$AA^1 = 2a$

3) The equation of the **Transverse axis** is Y = 0

4) The line segment BB^1 is the **Conjugate axis** of the hyperbola ,$BB^1 = 2b$

5) The equation of the **Conjugate axis** is X = 0

6) The point O is the intersection of **Transverse** and **Conjugate** axis.

7) The co-ordinates of O are (0,0)

8) The **foci** of the hyperbola are S(ae,0)and S^{I}(-ae,0)

9) The vertical lines passing through the focus are known as **Latusrectum**

10) The length of the Latusrectum is $\frac{2b^2}{a}$

11) The points A (a,0) and A^1(-a,0)

12) The **eccentricity** of the hyperbola is e = $\sqrt{1+\frac{b^2}{a^2}}$

13) The vertical lines $M_1M_1^1$ and $M_2M_2^1$ are known as the **directrix** of the hyperbola and their respective equations are x = $\frac{a}{e}$ and x = $\frac{-a}{e}$

DIFFERENTIATION

In all practical situations we come across a number of variables. The **variable** is one which takes different values, whereas a **constant** takes a fixed value.

Let x be the independent variable. That means x can take any value. Let y be a variable depending on the value of x. Then y is called the dependent variable. Then y is said to be a function of x and it is denoted by y = f(x)

For example if x denotes the time and y denotes the plant growth, then we know that the plant growth depends upon time. In that case, the function y=f(x) represents the growth function. The rate of change of y with respect to x is denoted by $\frac{dy}{dx}$ and called as the derivative of function y with respect to x.

S.No.	Form of Functions	y=f(x)	$\frac{dy}{dx}$
1.	Power Formula	x^n	$\frac{d(x^n)}{dx} = nx^{n-1}$
2.	Constant	C	0
3.	Constant with variable	Cy	$C\frac{dy}{dx}$
4.	Exponential	e^x	e^x
5.	Constant power x	a^x	$a^x \log a$
6.	Logirthamic	logx	$\frac{1}{x}$
7.	Differentiation of a sum	y = u + v where u and v are functions of x.	$\frac{dy}{dx} = \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$
8.	Differentiation of a difference	y = u – v where u and v are functions of x.	$\frac{dy}{dx} = \frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$
9.	Product rule of differentiation	y = uv, where u and v are functions of x.	$\frac{dy}{dx} = \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
10.	Quotient rule of differentiation	y = $\frac{u}{v}$,	$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

		where u and v are functions of x.	where $u' = \frac{du}{dx}$, $v' = \frac{dv}{dx}$

Example

1. Differentiate each of the following function $f(x) = 15x^{100} - 3x^{12} + 5x - 46$

Solution

$$f'(x) = 15(100)x^{99} - 3(12)x^{11} + 5(1)x^0 - 0$$
$$= 1500x^{99} - 36x^{11} + 5$$

2. Differentiate following function $g(t) = 2t^6 + 7t^{-6}$

Solution

Here is the derivative.

$$g'(t) = 2(6)t^5 + 7(-6)t^{-7}$$
$$= 12t^5 - 42t^{-7}$$

3. Differentiate following function $y = 8z^3 - \frac{1}{3z^5} + z - 23$

Solution

$$y = 8z^3 - \frac{1}{3}z^{-5} + z - 23$$

diff. w.r.to x

$$y' = 24z^2 + \frac{5}{3}z^{-6} + 1$$

4. Differentiate the following functions.

a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

Solution

$$y = \sqrt[3]{x^2}\left(2x - x^2\right)$$

$$y = x^{\frac{2}{3}}\left(2x - x^2\right)$$

diff y w. r. to x $y' = \frac{2}{3}x^{-\frac{1}{3}}\left(2x - x^2\right) + x^{\frac{2}{3}}\left(2 - 2x\right)$

5. Differentiate the following functions. $f\left(x\right) = \left(6x^3 - x\right)\left(10 - 20x\right)$

$$f\left(x\right) = \left(6x^3 - x\right)\left(10 - 20x\right)$$

diff f(x) w r to x

$$\begin{aligned} f'\left(x\right) &= \left(18x^2 - 1\right)\left(10 - 20x\right) + \left(6x^3 - x\right)\left(-20\right) \\ &= -480x^3 + 180x^2 + 40x - 10 \end{aligned}$$

Derivatives of the six trigonometric functions

$$\frac{d}{dx}\left(\sin\left(x\right)\right) = \cos\left(x\right) \qquad \frac{d}{dx}\left(\cos\left(x\right)\right) = -\sin\left(x\right)$$

$$\frac{d}{dx}\left(\tan\left(x\right)\right) = \sec^2\left(x\right) \qquad \frac{d}{dx}\left(\cot\left(x\right)\right) = -\csc^2\left(x\right)$$

$$\frac{d}{dx}\left(\sec\left(x\right)\right) = \sec\left(x\right)\tan\left(x\right) \qquad \frac{d}{dx}\left(\csc\left(x\right)\right) = -\csc\left(x\right)\cot\left(x\right)$$

Example

1. Differentiate each of the following functions.

$$g\left(x\right) = 3\sec\left(x\right) - 10\cos\left(x\right)$$

Solution We'll just differentiate each term using the formulas from above.

$$\begin{aligned} g'\left(x\right) &= 3\sec\left(x\right)\tan\left(x\right) - 10\left(-\sin\left(x\right)\right) \\ &= 3\sec\left(x\right)\tan\left(x\right) + 10\sin\left(x\right) \end{aligned}$$

2. Differentiate each of the following functions $y = 5\sin\left(x\right)\cot\left(x\right) + 4\csc\left(x\right)$

Here's the derivative of this function.

$$\begin{aligned} y' &= 5\cos\left(x\right)\cot\left(x\right) + 5\sin\left(x\right)\left(-\csc^2\left(x\right)\right) - 4\csc\left(x\right)\cot\left(x\right) \\ &= 5\cos\left(x\right)\cot\left(x\right) - 5\csc\left(x\right) - 4\csc\left(x\right)\cot\left(x\right) \end{aligned}$$

Note that in the simplification step we took advantage of the fact that

$$\csc(x) = \frac{1}{\sin(x)}$$

to simplify the second term a little.

3. Differentiate each of the following functions $P(t) = \dfrac{\sin(t)}{3 - 2\cos(t)}$

In this part we'll need to use the quotient rule.

$$\begin{aligned} P'(t) &= \frac{\cos(t)\left(3 - 2\cos(t)\right) - \sin(t)\left(2\sin(t)\right)}{\left(3 - 2\cos(t)\right)^2} \\ &= \frac{3\cos(t) - 2\cos^2(t) - 2\sin^2(t)}{\left(3 - 2\cos(t)\right)^2} \end{aligned}$$

CHAIN RULE DIFFERENTIATION

If y is a function of u ie y = f(u) and u is a function of x ie u = g(x) then y is related to x through the intermediate function u ie y =f(g(x))

$\therefore$ y is differentiable with respect to x

Furthermore, let y=f(g(x)) and u=g(x), then

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

There are a number of related results that also go under the name of "chain rules." For example, if y=f(u) u=g(v), and v=h(x),

then $$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}$$

Problem

Differentiate the following with respect to x

1. $y = (3x^2+4)^3$
2. $y = e^{x^{-2}}$

Marginal Analysis

Let us assume that the total cost C is represented as a function total output q. (i.e) C= f(q).

Then marginal cost is denoted by MC= $\frac{dc}{dq}$

The average cost = $\frac{TC}{Q}$

Similarly if U = u(x) is the utility function of the commodity x then

the marginal utility MU = $\frac{dU}{dx}$

The total revenue function TR is the product of quantity demanded Q and the price P per unit of that commodity then TR = Q.P = f(Q)

Then the marginal revenue denoted by MR is given by $\frac{dR}{dQ}$

The average revenue = $\frac{TR}{Q}$

Problem

1. If the total cost function is $C = Q^3 - 3Q^2 + 15Q$. Find Marginal cost and average cost.

Solution

MC = $\frac{dc}{dq}$

AC = $\frac{TC}{Q}$

2. The demand function for a commodity is P= (a - bQ). Find marginal revenue.
(the demand function is generally known as Average revenue function). Total revenue TR = P.Q = Q. (a - bQ) and marginal revenue MR= $\frac{d\left(aQ-bQ^2\right)}{dq}$

Growth rate and relative growth rate

The growth of the plant is usually measured in terms of dry mater production and as denoted by W. Growth is a function of time t and is denoted by W=g(t) it is called a growth function. Here t is the independent variable and w is the dependent variable.

The derivative $\frac{dw}{dt}$ is the growth rate (or) the absolute growth rate gr= $\frac{dw}{dt}$. GR = $\frac{dw}{dt}$

The relative growth rate i.e defined as the absolute growth rate divided by the total dry matter production and is denoted by RGR.

i.e RGR = $\frac{1}{w}.\frac{dw}{dt} = \frac{absolute\,growth\,rate}{total\,dry\,matter\,production}$

Problem

1. If G = at^2+b sin t +5 is the growth function function the growth rate and relative growth rate.

   GR = $\frac{dG}{dt}$

   RGR = $\frac{1}{G}.\frac{dG}{dt}$

Implicit Functions

If the variables x and y are related with each other such that f (x, y) = 0 then it is called Implicit function. A function is said to be **explicit** when one variable can be expressed completely in terms of the other variable.

For example, $y = x^3 + 2x^2 + 3x + 1$ is an Explicit function

$xy^2 + 2y + x = 0$ is an implicit function

Problem

For example, the implicit equation xy=1 can be solved by differentiating implicitly gives

$$\frac{d(xy)}{dx}=\frac{d(1)}{dx}$$

$$x\frac{dy}{dx}+y=0$$

$$\frac{dy}{dx}=-\frac{y}{x}.$$

Implicit differentiation is especially useful when y'(x)is needed, but it is difficult or inconvenient to solve for y in terms of x.

Example: Differentiate the following function with respect to x $x^3y^6+\mathbf{e}^{1-x}-\cos(5y)=y^2$

Solution

So, just differentiate as normal and tack on an appropriate derivative at each step. Note as well that the first term will be a product rule.

$$3x^2x'y^6+6x^3y^5y'-x'\mathbf{e}^{1-x}+5y'\sin(5y)=2yy'$$

Example: Find y' for the following function.

$$x^2+y^2=9$$

Solution

In this example we really are going to need to do implicit differentiation of *x* and write *y* as *y(x)*.

$$\frac{d}{dx}\left(x^2+\left[y(x)\right]^2\right)=\frac{d}{dx}(9)$$
$$2x+2\left[y(x)\right]^1y'(x)=0$$

Notice that when we differentiated the *y* term we used the chain rule.

Example: Find y' for the following. $x^3y^5+3x=8y^3+1$

Solution

First differentiate both sides with respect to *x* and notice that the first time on left side will be a product rule.

$$3x^2y^5+5x^3y^4y'+3=24y^2y'$$

Remember that very time we differentiate a *y* we also multiply that term by $y'y'$ since we are just using the chain rule. Now solve for the derivative.

$$3x^2y^5+3=24y^2y'-5x^3y^4y'$$
$$3x^2y^5+3=\left(24y^2-5x^3y^4\right)y'$$
$$y'=\frac{3x^2y^5+3}{24y^2-5x^3y^4}$$

The algebra in these can be quite messy so be careful with that.

Example:

Find y' for the following $x^2\tan(y)+y^{10}\sec(x)=2x$

Here we've got two product rules to deal with this time.

$$2x\tan(y)+x^2\sec^2(y)y'+10y^9y'\sec(x)+y^{10}\sec(x)\tan(x)=2$$

Notice the derivative tacked onto the secant. We differentiated a *y* to get to that point and so we needed to tack a derivative on.

Now, solve for the derivative.

$$\left(x^2\sec^2(y)+10y^9\sec(x)\right)y'=2-y^{10}\sec(x)\tan(x)-2x\tan(y)$$
$$y'=\frac{2-y^{10}\sec(x)\tan(x)-2x\tan(y)}{x^2\sec^2(y)+10y^9\sec(x)}$$

Logarithmic Differentiation

For some problems, first by taking logarithms and then differentiating, it is easier to find $\frac{dy}{dx}$. Such process is called Logarithmic differentiation.

(i) If the function appears as a product of many simple functions then by taking logarithm so that the product is converted into a sum. It is now easier to differentiate them.

(ii) If the variable x occurs in the exponent then by taking logarithm it is reduced to a familiar form to differentiate.

Example: Differentiate the function.

$$y=\frac{x^5}{(1-10x)\sqrt{x^2+2}}$$

Solution

Differentiating this function could be done with a product rule and a quotient rule. We can simplify things somewhat by taking logarithms of both sides.

$$\ln y = \ln\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right)$$

$$\ln y = \ln\left(x^5\right) - \ln\left((1-10x)\sqrt{x^2+2}\right)$$

$$\ln y = \ln\left(x^5\right) - \ln(1-10x) - \ln\left(\sqrt{x^2+2}\right)$$

$$\frac{y'}{y} = \frac{5x^4}{x^5} - \frac{-10}{1-10x} - \frac{\frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}(2x)}{\left(x^2+1\right)^{\frac{1}{2}}}$$

$$\frac{y'}{y} = \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+1}$$

Example

Differentiate $y = x^x$

Solution

First take the logarithm of both sides as we did in the first example and use the logarithm properties to simplify things a little.

$$\ln y = \ln x^x$$
$$\ln y = x \ln x$$

Differentiate both sides using implicit differentiation.

$$\frac{y'}{y} = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$$

As with the first example multiply by *y* and substitute back in for *y*.

$$y' = y(1+\ln x)$$
$$= x^x(1+\ln x)$$

PARAMETRIC FUNCTIONS

Sometimes variables x and y are expressed in terms of a third variable called **parameter.** We find $\frac{dy}{dx}$ without eliminating the third variable.

Let x = f(t) and y = g(t) then

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} = \frac{dy/dt}{dx/dt}$$

Problem

1. Find for the parametric function x =a cos θ , y = b sin θ

Solution

$$\frac{dx}{d\theta} = -a \sin \theta \qquad \frac{dy}{d\theta} = b \cos \theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{b \cos \theta}{-a \sin \theta}$$

$$= -\frac{b}{a} \cot \theta$$

Inference of the differentiation

Let y = f(x) be a given function then the first order derivative is $\frac{dy}{dx}$.

The geometrical meaning of the first order derivative is that it represents the slope of the curve y = f(x) at x.

The physical meaning of the first order derivative is that it represents the rate of change of y with respect to x.

PROBLEMS ON HIGHER ORDER DIFFERENTIATION

The rate of change of y with respect x is denoted by $\frac{dy}{dx}$ and called as the first order derivative of function y with respect to x.

The first order derivative of y with respect to x is again a function of x, which again be differentiated with respect to x and it is called second order derivative of y = f(x) and is denoted by $\frac{d^2y}{dx^2}$ which is equal to $\frac{d}{dx}\left(\frac{dy}{dx}\right)$

In the similar way higher order differentiation can be defined. Ie. The nth order derivative of y=f(x) can be obtained by differentiating n-1^{th} derivative of y=f(x)

$\frac{d^n y}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$ where n= 2,3,4,5….

Problem

Find the first, second and third derivative of

1. y = e^{ax+b}
2. y = log(a-bx)
3. y = sin (ax+b)

Partial Differentiation

So far we considered the function of a single variable y = f(x) where x is the only independent variable. When the number of independent variable exceeds one then we call it as the function of several variables.

Example

z = f(x,y) is the function of two variables x and y , where x and y are independent variables.

U=f(x,y,z) is the function of three variables x,y and z , where x, y and z are independent variables.

In all these functions there will be only one dependent variable.

Consider a function z = f(x,y). The partial derivative of z with respect to x denoted by $\frac{\partial z}{\partial x}$ and is obtained by differentiating z with respect to x keeping y as a constant. Similarly the partial derivative of z with respect to y denoted by $\frac{\partial z}{\partial y}$ and is obtained by differentiating z with respect to y keeping x as a constant.

Problem

1. Differentiate U = log (ax+by+cz) partially with respect to x, y & z

We can also find higher order partial derivatives for the function z = f(x,y) as follows

(i) The second order partial derivative of z with respect to x denoted as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}$ is obtained by partially differentiating $\frac{\partial z}{\partial x}$ with respect to x. this is also known as direct second order partial derivative of z with respect to x.

(ii)The second order partial derivative of z with respect to y denoted as $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}$ is obtained by partially differentiating $\frac{\partial z}{\partial y}$ with respect to y this is also known as direct second order partial derivative of z with respect to y

(iii) The second order partial derivative of z with respect to x and then y denoted as $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial y \partial x}$ is obtained by partially differentiating $\frac{\partial z}{\partial x}$ with respect to y. this is also known as mixed second order partial derivative of z with respect to x and then y

iv) The second order partial derivative of z with respect to y and then x denoted as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x \partial y}$ is obtained by partially differentiating $\frac{\partial z}{\partial y}$ with respect to x. this is also known as mixed second order partial derivative of z with respect to y and then x. In similar way higher order partial derivatives can be found.

Problem

Find all possible first and second order partial derivatives of

1) $z = \sin(ax + by)$

2) $u = xy + yz + zx$

Homogeneous Function

A function in which each term has the same degree is called a homogeneous function.

Example

1) $x^2 - 2xy + y^2 = 0 \rightarrow$ homogeneous function of degree 2.
2) $3x + 4y = 0$ $\rightarrow$ homogeneous function of degree 1.
3) $x^3 + 3x^2y + xy^2 - y^3 = 0 \rightarrow$ homogeneous function of degree 3.

To find the degree of a homogeneous function we proceed as follows:

Consider the function f(x,y) replace x by tx and y by ty if $f(tx, ty) = t^n f(x, y)$ then n gives the degree of the homogeneous function. This result can be extended to any number of variables.

Problem

Find the degree of the homogeneous function

1. $f(x, y) = x^2 - 2xy + y^2$

2. $f(x,y) = \dfrac{x-y}{x+y}$

Euler's theorem on homogeneous function

If U= f(x,y,z) is a homogeneous function of degree n in the variables x, y & z then

$$x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n.u$$

Problem

Verify Euler's theorem for the following function

1. $u(x,y) = x^2 - 2xy + y^2$
2. $u(x,y) = x^3 + y^3 + z^3 - 3xyz$

INCREASING AND DECREASING FUNCTION

Increasing function

A function y= f(x) is said to be an increasing function if $f(x_1) < f(x_2)$ for all $x_1 < x_2$.

The condition for the function to be increasing is that its first order derivative is always greater than zero .

i.e $\dfrac{dy}{dx} > 0$

Decreasing function

A function y= f(x) is said to be a decreasing function if $f(x_1) > f(x_2)$ for all $x_1 < x_2$.

The condition for the function to be decreasing is that its first order derivative is always less than zero .

i.e $\dfrac{dy}{dx} < 0$

Problems

1. Show that the function $y = x^3 + x$ is increasing for all x.
2. Find for what values of x is the function $y = 8 + 2x - x^2$ is increasing or decreasing ?

Maxima and Minima Function of a single variable

A function y = f(x) is said to have maximum at x = a if f(a) > f(x) in the neighborhood of the point x = a and f(a) is the maximum value of f(x) . The point x = a is also known as local maximum point.

A function y = f(x) is said to have minimum at x = a if f(a) < f(x) in the neighborhood of the point x = a and f(a) is the minimum value of f(x) . The point x = a is also known as local minimum point.

The points at which the function attains maximum or minimum are called the turning points or stationary points

A function y=f(x) can have more than one **maximum or minimum point** . Maximum of all the maximum points is called **Global maximum** and minimum of all the minimum points is called **Global minimum**.

A point at which neither maximum nor minimum is called **Saddle point**.

[Consider a function y = f(x). If the function increases upto a particular point x = a and then decreases it is said to have a maximum at x = a. If the function decreases upto a point x = b and then increases it is said to have a minimum at a point x=b.]

The necessary and the sufficient condition for the function y=f(x) to have a maximum or minimum can be tabulated as follows

	Maximum	**Minimum**
First order or necessary condition	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} =0$
Second order or sufficient condition	$\frac{d^2y}{dx^2} < 0$	$\frac{d^2y}{dx^2} > 0$

Working Procedure

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

2. Equate $\frac{dy}{dx}$ =0 and solve for x. this will give the turning points of the function.

3. Consider a turning point x = a then substitute this value of x in $\frac{d^2y}{dx^2}$ and find the nature of the second derivative. If $\left(\frac{d^2y}{dx^2}\right)_{at\, x=a}$ < *0*, then the function has a maximum

value at the point x = a. If $\left(\frac{d^2y}{dx^2}\right)_{at\,x=a}$ > *0*, then the function has a minimum value at the point x = a.

4. Then substitute x = a in the function y = f(x) that will give the maximum or minimum value of the function at x = a.

Problem

Find the maximum and minimum values of the following function

1. $y = x^3 - 3x + 1$

Differential Equations

Differential equation is an equation in which differential coefficients occur.

A differential equation is of two types

(1) Ordinary differential equation

(2) Partial differential equation

An ordinary differential equation is one which contains a single independent variable.

Example:

$$\frac{dy}{dx} = 2\sin x \qquad \frac{dy}{dx} + 4y = e^x$$

A partial differential equation is one containing more than one independent variable.

Examples

1. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$

2. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

Here we deal with only ordinary differential equations.

Definitions

Order

The order of a differential equation is the order of the highest order derivative appearing in it.

$\frac{dy}{dx} + 4y = e^x$ Order 1

$\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^2 - 3y = 0$ Order -2

Degree

The degree of a differential equation is defined as the degree of highest ordered derivative occurring in it after removing the radical sign.

Example

Give the degree and order of the following differential equation.

1) $5(x+y)\frac{dy}{dx} + 3xy = x^2$ degree -1, order -1

2) $\left(\frac{d^2y}{dx^2}\right)^3 - 6\frac{dy}{dx} + xy = 20$ degree -3, order -2

3) $\sqrt{1+\frac{d^2y}{dx^2}} = 3\frac{dy}{dx}+1$

Squaring on both sides

$$1+\frac{d^2y}{dx^2} = 9\left(\frac{dy}{dx}\right)^2 +6\frac{dy}{dx}+1$$

degree -1, order 2

4) $\left(1+\frac{dy}{dx}\right)^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$

$1+3\frac{dy}{dx}+3\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dx}\right)^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$

degree – 2, order – 2

Note

If the degree of the differential equations is one. It is called a linear differential equation.

Formation of differential equations

Given the solution of differential equation, we can form the corresponding differential equation. Suppose the solution contains one arbitrary constant then differentiate the solution once with respect to x and eliminating the arbitrary constant from the two equations. We get the required equation. Suppose the solution contains two arbitrary constant then differentiate the solution twice with respect to x and eliminating the arbitrary constant between the three equations.

Solution of differential equations

(i) Variable separable method,

(ii) Homogenous differential equation

iii) Linear differential equation

Variable separable method

Consider a differential equation $\frac{dy}{dx} = f(x)$

Here we separate the variables in such a way that we take the terms containing variable x on one side and the terms containing variable y on the other side. Integrating we get the solution.

Note

The following formulae are useful in solving the differential equations

(i) $d(xy) = xdy + ydx$

(ii) $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

(iii) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

Homogenous differential equation

Consider a differential equation of the form

$$\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)} \longrightarrow \text{(i)}$$

where f_1 and f_2 are homogeneous functions of same degree in x and y.

Here put $y = vx$

$$\left.\begin{aligned} & y = vx \\ & \therefore \frac{dy}{dx} = v + x\frac{dv}{dx} \end{aligned}\right\} \longrightarrow \text{(ii)}$$

Substitute equation(ii) in equation (i) it reduces to a differential equation in the variables v and x. Separating the variables and integrating we can find the solution.

Linear differential equation

A linear differential equation of the first order is of the form $\frac{dy}{dx} + py = Q$, Where p and Q are functions of x only.

To solve this equation first we find the integrating factor given by

Integrating factor = I.F = $e^{\int pdx}$

Then the solution is given by

$$ye^{\int pdx} = \int Q e^{\int pdx} dx + c$$ where c is an arbitrary constant.

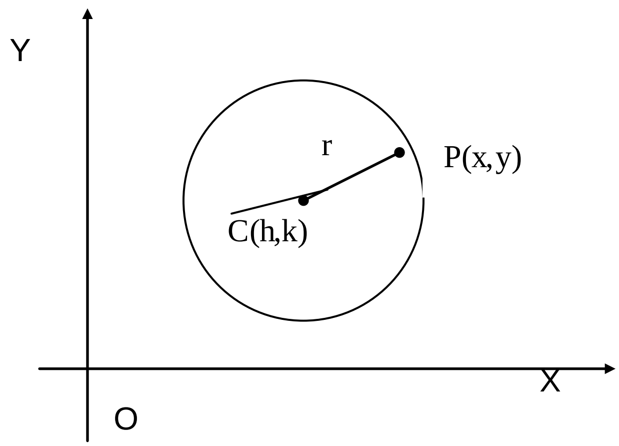
Circles

A circle is defined as the locus of the point, which moves in such a way, that its distance from a fixed point is always constant. The fixed point is called **centre** of the circle and the constant distance is called the **radius** of the circle.

The equation of the circle when the centre and radius are given

Let C (h,k) be the centre and r be the radius of the circle. Let P(x,y) be any point on the circle.

$CP = r \Rightarrow CP^2 = r^2 \Rightarrow (x-h)^2 + (y-k)^2 = r^2$ is the required equation of the circle.



Note :

If the center of the circle is at the origin i.e., C(h,k)=(0,0) then the equation of the circle is $x^2 + y^2 = r^2$

<u>The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$</u>

Consider the equation $x^2 + y^2 + 2gx + 2fy + c = 0$. This can be written as

$$x^2 + y^2 + 2gx + 2fy + g^2 + f^2 = g^2 + f^2 - c$$

(i.e) $$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c$$

$$(x + g)^2 + (y + f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

$$[x - (-g)]^2 + [y - (-f)]^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

This is of the form $(x-h)^2 + (y-k)^2 = r^2$

$\therefore$ The considered equation represents a circle with centre (-g,-f) and radius $\sqrt{g^2 + f^2 - c}$

$\therefore$ The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

where

c = The Center of the circle whose coordinates are (-g,-f)

r = The radius of the circle = $\sqrt{g^2 + f^2 - c}$

Note

The general second degree equation

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

Represents a circle if

(i) $a = b$ i.e., coefficient of x^2 = coefficient of y^2

(ii) $h = 0$ i.e., no xy term

CHAIN RULE DIFFERENTIATION

If y is a function of u ie y = f(u) and u is a function of x ie u = g(x) then y is related to x through the intermediate function u ie y =f(g(x))

$\therefore$ y is differentiable with respect to x

Furthermore, let y=f(g(x)) and u=g(x), then

$$\frac{dy}{dx} = \frac{dy}{du}\ \frac{du}{dx}$$

There are a number of related results that also go under the name of "chain rules." For example, if y=f(u) u=g(v), and v=h(x),

then $$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}$$

Problem

Differentiate the following with respect to x

1. $y = (3x^2+4)^3$
2. $y = e^{x^{-2}}$

Marginal Analysis

Let us assume that the total cost C is represented as a function total output q. (i.e) C= f(q).

Then marginal cost is denoted by MC= $\frac{dc}{dq}$

The average cost = $\frac{TC}{Q}$

Similarly if U = u(x) is the utility function of the commodity x then

the marginal utility MU = $\frac{dU}{dx}$

The total revenue function TR is the product of quantity demanded Q and the price P per unit of that commodity then TR = Q.P = f(Q)

Then the marginal revenue denoted by MR is given by $\frac{dR}{dQ}$

The average revenue = $\frac{TR}{Q}$

Problem

1. If the total cost function is $C = Q^3 - 3Q^2 + 15Q$. Find Marginal cost and average cost.

Solution

MC = $\frac{dc}{dq}$

AC = $\frac{TC}{Q}$

2. The demand function for a commodity is P= (a - bQ). Find marginal revenue.

(the demand function is generally known as Average revenue function). Total revenue TR = P.Q = Q. (a - bQ) and marginal revenue MR= $\frac{d\left(aQ - bQ^2\right)}{dq}$

Growth rate and relative growth rate

The growth of the plant is usually measured in terms of dry mater production and as denoted by W. Growth is a function of time t and is denoted by W=g(t) it is called a growth function. Here t is the independent variable and w is the dependent variable.

The derivative $\frac{dw}{dt}$ is the growth rate (or) the absolute growth rate gr= $\frac{dw}{dt}$. GR = $\frac{dw}{dt}$

The relative growth rate i.e defined as the absolute growth rate divided by the total dry matter production and is denoted by RGR.

i.e RGR = $\frac{1}{w} . \frac{dw}{dt} = \frac{absolute\,growth\,rate}{total\,dry\,matter\,production}$

Problem

1. If G = at^2+b sin t +5 is the growth function function the growth rate and relative growth rate.

   GR = $\frac{dG}{dt}$

   RGR = $\frac{1}{G} . \frac{dG}{dt}$

Implicit Functions

If the variables x and y are related with each other such that f (x, y) = 0 then it is called Implicit function. A function is said to be **explicit** when one variable can be expressed completely in terms of the other variable.

For example, $y = x^3 + 2x^2 + 3x + 1$ is an Explicit function

$xy^2 + 2y + x = 0$ is an implicit function

Problem

For example, the implicit equation xy=1 can be solved by differentiating implicitly gives

$$\frac{d(xy)}{dx}=\frac{d(1)}{dx}$$

$$x\frac{dy}{dx}+y=0$$

$$\frac{dy}{dx}=-\frac{y}{x}.$$

Implicit differentiation is especially useful when y'(x)is needed, but it is difficult or inconvenient to solve for y in terms of x.

Example: *Differentiate the following function with respect to x* $x^3y^6+\mathbf{e}^{1-x}-\cos(5y)=y^2$

Solution

So, just differentiate as normal and tack on an appropriate derivative at each step. Note as well that the first term will be a product rule.

$$3x^2x'y^6+6x^3y^5y'-x'\mathbf{e}^{1-x}+5y'\sin(5y)=2yy'$$

Example: Find y' for the following function.

$$x^2+y^2=9$$

Solution

In this example we really are going to need to do implicit differentiation of *x* and write *y* as *y(x)*.

$$\frac{d}{dx}\left(x^2+\left[y(x)\right]^2\right)=\frac{d}{dx}(9)$$

$$2x+2\left[y(x)\right]^1y'(x)=0$$

Notice that when we differentiated the *y* term we used the chain rule.

Example: Find y' for the following. $x^3y^5+3x=8y^3+1$

Solutio: First differentiate both sides with respect to *x* and notice that the first time on left side will be a product rule.

$$3x^2y^5+5x^3y^4y'+3=24y^2y'$$

Remember that very time we differentiate a *y* we also multiply that term by $y'y'$ since we are just using the chain rule. Now solve for the derivative.

$$3x^2y^5 + 3 = 24y^2y' - 5x^3y^4y'$$
$$3x^2y^5 + 3 = \left(24y^2 - 5x^3y^4\right)y'$$
$$y' = \frac{3x^2y^5 + 3}{24y^2 - 5x^3y^4}$$

The algebra in these can be quite messy so be careful with that.

Example: Find y' for the following $x^2\tan(y) + y^{10}\sec(x) = 2x$

Here we've got two product rules to deal with this time.

$$2x\tan(y) + x^2\sec^2(y)\,y' + 10y^9y'\sec(x) + y^{10}\sec(x)\tan(x) = 2$$

Notice the derivative tacked onto the secant. We differentiated a *y* to get to that point and so we needed to tack a derivative on.

Now, solve for the derivative.

$$\left(x^2\sec^2(y) + 10y^9\sec(x)\right)y' = 2 - y^{10}\sec(x)\tan(x) - 2x\tan(y)$$
$$y' = \frac{2 - y^{10}\sec(x)\tan(x) - 2x\tan(y)}{x^2\sec^2(y) + 10y^9\sec(x)}$$

Logarithmic Differentiation

For some problems, first by taking logarithms and then differentiating, it is easier to find $\frac{dy}{dx}$. Such process is called Logarithmic differentiation.

(i) If the function appears as a product of many simple functions then by taking logarithm so that the product is converted into a sum. It is now easier to differentiate them.

(ii) If the variable x occurs in the exponent then by taking logarithm it is reduced to a familiar form to differentiate.

Example

Differentiate the function.

$$y = \frac{x^5}{(1-10x)\sqrt{x^2+2}}$$

Solution

Differentiating this function could be done with a product rule and a quotient rule. We can simplify things somewhat by taking logarithms of both sides.

$$\ln y = \ln\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right)$$

$$\ln y = \ln\left(x^5\right) - \ln\left((1-10x)\sqrt{x^2+2}\right)$$

$$\ln y = \ln\left(x^5\right) - \ln(1-10x) - \ln\left(\sqrt{x^2+2}\right)$$

$$\frac{y'}{y} = \frac{5x^4}{x^5} - \frac{-10}{1-10x} - \frac{\frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}(2x)}{\left(x^2+1\right)^{\frac{1}{2}}}$$

$$\frac{y'}{y} = \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+1}$$

Example

Differentiate $y = x^x$

Solution

First take the logarithm of both sides as we did in the first example and use the logarithm properties to simplify things a little.

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

Differentiate both sides using implicit differentiation.

$$\frac{y'}{y} = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$$

As with the first example multiply by *y* and substitute back in for *y*.

$$y' = y(1+\ln x)$$

$$= x^x(1+\ln x)$$

PARAMETRIC FUNCTIONS

Sometimes variables x and y are expressed in terms of a third variable called **parameter.** We find $\frac{dy}{dx}$ without eliminating the third variable.

Let x = f(t) and y = g(t) then

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} = \frac{dy/dt}{dx/dt}$$

Problem

1. Find for the parametric function x =a cos θ , y = b sin θ

Solution

$$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{b\cos\theta}{-a\sin\theta}$$

$$= -\frac{b}{a}\cot\theta$$

Inference of the differentiation

Let y = f(x) be a given function then the first order derivative is $\frac{dy}{dx}$.

The geometrical meaning of the first order derivative is that it represents the slope of the curve y = f(x) at x.

The physical meaning of the first order derivative is that it represents the rate of change of y with respect to x.

PROBLEMS ON HIGHER ORDER DIFFERENTIATION

The rate of change of y with respect x is denoted by $\frac{dy}{dx}$ and called as the first order derivative of function y with respect to x.

The first order derivative of y with respect to x is again a function of x, which again be differentiated with respect to x and it is called second order derivative of y = f(x) and is denoted by $\frac{d^2y}{dx^2}$ which is equal to $\frac{d}{dx}\left(\frac{dy}{dx}\right)$

In the similar way higher order differentiation can be defined. Ie. The nth order derivative of y=f(x) can be obtained by differentiating n-1th derivative of y=f(x)

$$\frac{d^n y}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1} y}{dx^{n-1}}\right)$$ where n= 2,3,4,5….

Problem

Find the first , second and third derivative of

1. y = e^{ax+b}
2. y = log(a-bx)
3. y = sin (ax+b)

Partial Differentiation

So far we considered the function of a single variable y = f(x) where x is the only independent variable. When the number of independent variable exceeds one then we call it as the function of several variables.

Example

z = f(x,y) is the function of two variables x and y , where x and y are independent variables.

U=f(x,y,z) is the function of three variables x,y and z , where x, y and z are independent variables.

In all these functions there will be only one dependent variable.

Consider a function z = f(x,y). The partial derivative of z with respect to x denoted by $\frac{\partial z}{\partial x}$ and is obtained by differentiating z with respect to x keeping y as a constant. Similarly the partial derivative of z with respect to y denoted by $\frac{\partial z}{\partial y}$ and is obtained by differentiating z with respect to y keeping x as a constant.

Problem

1. Differentiate U = log (ax+by+cz) partially with respect to x, y & z

We can also find higher order partial derivatives for the function z = f(x,y) as follows

(i) The second order partial derivative of z with respect to x denoted as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$ is obtained by partially differentiating $\frac{\partial z}{\partial x}$ with respect to x. this is also known as direct second order partial derivative of z with respect to x.

(ii)The second order partial derivative of z with respect to y denoted as $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}$ is obtained by partially differentiating $\frac{\partial z}{\partial y}$ with respect to y this is also known as direct second order partial derivative of z with respect to y

(iii) The second order partial derivative of z with respect to x and then y denoted as $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial y \partial x}$ is obtained by partially differentiating $\frac{\partial z}{\partial x}$ with respect to y. this is also known as mixed second order partial derivative of z with respect to x and then y

iv) The second order partial derivative of z with respect to y and then x denoted as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x \partial y}$ is obtained by partially differentiating $\frac{\partial z}{\partial y}$ with respect to x. this is also known as mixed second order partial derivative of z with respect to y and then x. In similar way higher order partial derivatives can be found.

Problem

Find all possible first and second order partial derivatives of

1) z = sin(ax +by)

2) u = xy + yz + zx

Homogeneous Function

A function in which each term has the same degree is called a homogeneous function.

Example

1) $x^2 - 2xy + y^2 = 0 \rightarrow$ homogeneous function of degree 2.
2) $3x + 4y = 0 \rightarrow$ homogeneous function of degree 1.
3) $x^3 + 3x^2y + xy^2 - y^3 = 0 \rightarrow$ homogeneous function of degree 3.

To find the degree of a homogeneous function we proceed as follows.

Consider the function f(x,y) replace x by tx and y by ty if $f(tx, ty) = t^n f(x, y)$ then n gives the degree of the homogeneous function. This result can be extended to any number of variables.

Problem

Find the degree of the homogeneous function

1. $f(x, y) = x^2 - 2xy + y^2$

2. $f(x,y) = \frac{x-y}{x+y}$

Euler's theorem on homogeneous function

If U= f(x,y,z) is a homogeneous function of degree n in the variables x, y & z then

$$x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n.u$$

Problem

Verify Euler's theorem for the following function

1. $u(x,y) = x^2 - 2xy + y^2$
2. $u(x,y) = x^3 + y^3 + z^3 - 3xyz$

INCREASING AND DECREASING FUNCTION

Increasing function

A function y= f(x) is said to be an increasing function if $f(x_1) < f(x_2)$ for all $x_1 < x_2$.

The condition for the function to be increasing is that its first order derivative is always greater than zero .

i.e $\frac{dy}{dx} > 0$

Decreasing function

A function y= f(x) is said to be a decreasing function if $f(x_1) > f(x_2)$ for all $x_1 < x_2$.

The condition for the function to be decreasing is that its first order derivative is always less than zero .

i.e $\frac{dy}{dx} < 0$

Problems

1. Show that the function $y = x^3 + x$ is increasing for all x.
2. Find for what values of x is the function $y = 8 + 2x - x^2$ is increasing or decreasing ?

Maxima and Minima Function of a single variable

A function y = f(x) is said to have maximum at x = a if f(a) > f(x) in the neighborhood of the point x = a and f(a) is the maximum value of f(x) . The point x = a is also known as local maximum point.

A function y = f(x) is said to have minimum at x = a if f(a) < f(x) in the neighborhood of the point x = a and f(a) is the minimum value of f(x) . The point x = a is also known as local minimum point.

The points at which the function attains maximum or minimum are called the turning points or stationary points

A function y=f(x) can have more than one **maximum or minimum point** . Maximum of all the maximum points is called **Global maximum** and minimum of all the minimum points is called **Global minimum**.

A point at which neither maximum nor minimum is called **Saddle point**.

[Consider a function y = f(x). If the function increases upto a particular point x = a and then decreases it is said to have a maximum at x = a. If the function decreases upto a point x = b and then increases it is said to have a minimum at a point x=b.]

The necessary and the sufficient condition for the function y=f(x) to have a maximum or minimum can be tabulated as follows

	Maximum	**Minimum**
First order or necessary condition	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 0$
Second order or sufficient condition	$\frac{d^2y}{dx^2} < 0$	$\frac{d^2y}{dx^2} > 0$

Working Procedure

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

2. Equate $\frac{dy}{dx}$ =0 and solve for x. this will give the turning points of the function.

3. Consider a turning point x = a then substitute this value of x in $\frac{d^2y}{dx^2}$ and find the nature of the second derivative. If $\left(\frac{d^2y}{dx^2}\right)_{at\,x=a}$ < *0*, then the function has a maximum value at the point x = a. If $\left(\frac{d^2y}{dx^2}\right)_{at\,x=a}$ > *0*, then the function has a minimum value at the point x = a.

4. Then substitute x = a in the function y = f(x) that will give the maximum or minimum value of the function at x = a.

Problem

Find the maximum and minimum values of the following function

1. $y = x^3 - 3x + 1$

INTEGRATION

Integration is a process, which is a inverse of differentiation. As the symbol $\frac{d}{dx}$ represents differentiation with respect to x, the symbol $\int dx$ stands for integration with respect to x.

Definition

If $\frac{d}{dx}[f(x)] = F(x)$ then f(x) is called the integral of F(x) denoted by $\int F(x)dx = f(x) + c$. This can be read it as integral of F(x) with respect to x is f(x) + c where c is an arbitrary constant. The integral $\int F(x)dx$ is known as **Indefinite integral and the function F(x) as integrand.**

Formula on integration

1). $\int x^n dx = \frac{x^{n+1}}{n+1}$ **+c** (n $\neq$-1)

2). $\int \frac{1}{x} dx = \log x$ **+c**

3). $\int dx$**= x+c**

4). $\int a^x dx = \frac{a^x}{\log a}$**+c**

5). $\int e^x$ dx = e^x **+c**

6). $\int (u(x) + v(x))dx = \int u(x)dx + \int v(x)dx$

7). $\int (c_1 u(x) \pm c_2 v(x))dx = \int c_1 u(x)dx \pm \int c_2 v(x)dx$

8). $\int cdx$**=** c x + d

9). $\int \sin x\, dx = -\cos x$**+c**

10). $\int \cos x\, dx = \sin x$ **+c**

11). $\int \sec^2 x dx = \tan x$ **+c**

12). $\int \cos ec^2 x dx = -\cot x$ **+c**

13). $\int \sec x \tan x \, dx = \sec x + c$

14). $\int \cos ecx \cot x \, dx = -\cos ec \, x + c$

13). $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$ **+c**

14). $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \frac{a + x}{a - x}$ **+c**

15). $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \frac{x - a}{x + a}$ **+c**

16). $\int \frac{1}{1 - x^2} dx = \tan^{-1} x + c$

Definite integral

If f(x) is indefinite integral of F(x) with respect to x then the Integral $\int_a^b F(x) dx$ is called definite integral of F(x) with respect to x from x = a to x = b. Here a is called the Lower limit and b is called the Upper limit of the integral.

$\int_a^b F(x) dx$ = $[f(x)]_a^b$ = f(Upper limit) - f(Lower limit)

= f(b) - f(a)

Note

While evaluating a definite integral no constant of integration is to be added. That is a definite integral has a definite value.

Method of substitution

Method –1

Formulae for the functions involving (ax + b)

Consider the integral

I = $\int (ax+b)^n dx$ -------------(1)

Where a and b are constants

Put a x + b = y

Differentiating with respect to x

a dx + 0 = dy

$$dx = \frac{dy}{a}$$

Substituting in (1)

I = $\int y^n . \frac{1}{a} dy$ +c

$= \frac{1}{a} \int y^n .dy$ +c

$= \frac{1}{a} \frac{y^{n+1}}{n+1}$ +c

$= \frac{1}{a} \left[\frac{(ax+b)^{n+1}}{n+1} \right]$ + c

Similarly this method can be applied for other formulae also.

Method II

Integrals of the functions of the form

$$\int f(x^n) x^{n-1} dx$$

put x^n =y,

$$nx^{n-1} = \frac{dy}{dx}$$

$$x^{n-1} dx = \frac{dy}{n}$$

Substituting we get

I = $\int f(y) \frac{dy}{n}$ and this can be integrated.

Method –III

Integrals of function of the type

$$\int [f(x)]^n f^1(x)dx$$

when n ≠ -1, put f(x) = y then $f^1(x)dx = dy$

$$\therefore \int [f(x)]^n f^1(x)dx = \int y^n \, dy$$

$$= \frac{y^{n+1}}{n+1}$$

$$= \frac{[f(x)]^{n+1}}{n+1}$$

when n= -1, the integral reduces to

$$\frac{f^1(x)}{f(x)}dx$$

putting y = f(x) then dy = f^1(x) dx

$$\therefore \int \frac{dy}{y} = \log y = \log f(x)$$

Method IV

Method of Partial Fractions

Integrals of the form $\int \frac{dx}{ax^2 + bx + c}$

Case.1

If denominator can be factorized into linear factors then we write the integrand as the sum or difference of two linear factors of the form

$$\frac{1}{(ax^2 + bx + c} = \frac{1}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$$

Case-2

In the given integral $\int \frac{dx}{ax^2 + bx + c}$ the denominator $ax^2 + bx + c$ can not be factorized into linear factors, then express $ax^2 + bx + c$ as the sum or difference of two perfect squares and then apply the formulae

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \frac{a+x}{a-x}$$

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \frac{x-a}{x+a}$$

Integrals of the form $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$

Write denominator as the sum or difference of two perfect squares

$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \int \frac{dx}{\sqrt{x^2 + a^2}}$ **or** $\int \frac{dx}{\sqrt{x^2 - a^2}}$ **or** $\int \frac{dx}{\sqrt{a^2 - x^2}}$

and then apply the formula

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log(x + \sqrt{x^2 + a^2})$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log(x + \sqrt{x^2 - a^2})$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

Integration by parts

If the given integral is of the form $\int udv$ then this can not be solved by any of techniques studied so far. To solve this integral we first take the product rule on differentiation

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides we get

$$\int \frac{d(uv)}{dx} dx = \int \left(u\frac{dv}{dx} + v\frac{du}{dx}\right)dx$$

then we have $uv = \int vdu + \int udv$

re arranging the terms we get

$\int udv = uv - \int vdu$ This formula is known **as integration by parts formula**

Select the functions u and dv appropriately in such a way that integral $\int vdu$ can be more easily integrable than the given integral

APPLICATION OF INTEGRATION

The area bounded by the function y=f(x), x=axis and the ordinates at x=a x=b is given by $A = \int_a^b f(x)dx$

INVERSE OF A MATRIX

Definition

Let A be any square matrix. If there exists another square matrix B Such that AB = BA = I (I is a unit matrix) then B is called the inverse of the matrix A and is denoted by A^{-1}.

The cofactor method is used to find the inverse of a matrix. Using matrices, the solutions of simultaneous equations are found.

Working Rule to find the inverse of the matrix

Step 1: Find the determinant of the matrix.

Step 2: If the value of the determinant is non zero proceed to find the inverse of the matrix.

Step 3: Find the cofactor of each element and form the cofactor matrix.

Step 4: The transpose of the cofactor matrix is the adjoint matrix.

Step 5: The inverse of the matrix $A^{-1} = \dfrac{adj(A)}{|A|}$

Example

Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$

Solution

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$

Step 1

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1(18-12) - 1(9-3) + (4-2)$$

$$= 6 - 6 + 2 = 2 \neq 0$$

Step 2

The value of the determinant is non zero

$\therefore A^{-1}$ exists.

Step 3

Let A_{ij} denote the cofactor of a_{ij} in $|A|$

$$A_{11} = Cofactor\ of\ 1 = (-1)^{1+1}\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} = 18 - 12 = 6$$

$$A_{12} = Cofactor\ of\ 1 = (-1)^{1+3}\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = -(9-3) = -6$$

$$A_{13} = Cofactor\ of\ 1 = (-1)^{1+3}\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 4 - 2 = 2$$

$$A_{21} = Cofactor\ of\ 1 = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -(9-4) = -5$$

$$A_{22} = Cofactor\ of\ 2 = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = 9 - 1 = 8$$

$$A_{23} = Cofactor\ of\ 3 = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -(4-1) = -3$$

$$A_{31} = Cofactor\ of\ 1 = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1$$

$$A_{32} = Cofactor\ of\ 4 = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -(3-1) = -2$$

$$A_{33} = Cofactor\ of\ 9 = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1$$

Step 4

The matrix formed by cofactors of element of determinant $|A|$ is $\begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$

$$\therefore \text{adj A} = \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

Step 5

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{2}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & \frac{-5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & \frac{-3}{2} & \frac{1}{2} \end{pmatrix}$$

SOLUTION OF LINEAR EQUATIONS

Let us consider a system of linear equations with three unknowns

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

The matrix form of the equation is AX=B where $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is a 3x3 matrix

X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B = $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Here AX = B

Pre multiplying both sides by A^{-1}.

$(A^{-1} A)X = A^{-1}B$

We know that $A^{-1} A = A A^{-1} = I$

$\therefore$ I X= $A^{-1}B$

since IX = X

Hence the solution is X = $A^{-1}B$.

Example

Solve the *x* + *y* + *z* = *1,* 3*x* + 5*y* + 6*z* = *4,* 9*x* + 26*y* + 36*z* =*16* by matrix method.

Solution

The given equations are *x* + *y* + *z* = *1,*

3*x* + 5*y* + 6*z* = *4,*

9*x* + 26*y* + 36*z* =*16*

Let A= $\begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ 9 & 26 & 36 \end{pmatrix}$, X= $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, B= $\begin{pmatrix} 1 \\ 4 \\ 16 \end{pmatrix}$

The given system of equations can be put in the form of the matrix equation AX=B

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ 9 & 26 & 36 \end{vmatrix} = 1(180-156) - 1(108-54) + 1(78-45)$$

$$= 24 - 54 + 33 = 3 \neq 0$$

The value of the determinant is non zero

$\therefore$ A^{-1} exists.

Let A_{ij} (i, j = 1,2,3) denote the cofactor of a_{ij} in $|A|$

$$A_{11} = Cofactor\ of\ 1 = (-1)^{1+1}\begin{vmatrix}5 & 6\\ 26 & 36\end{vmatrix} = 180-156 = 24$$

$$A_{12} = Cofactor\ of\ 1 = (-1)^{1+3}\begin{vmatrix}3 & 6\\ 9 & 36\end{vmatrix} = -(108-54) = -54$$

$$A_{13} = Cofactor\ of\ 1 = (-1)^{1+3}\begin{vmatrix}3 & 5\\ 9 & 26\end{vmatrix} = 78-45 = 33$$

$$A_{21} = Cofactor\ of\ 3 = (-1)^{2+1}\begin{vmatrix}1 & 1\\ 26 & 36\end{vmatrix} = -(36-26) = -10$$

$$A_{22} = Cofactor\ of\ 5 = (-1)^{2+2}\begin{vmatrix}1 & 1\\ 9 & 36\end{vmatrix} = 36-9 = 27$$

$$A_{23} = Cofactor\ of\ 6 = (-1)^{2+3}\begin{vmatrix}1 & 1\\ 9 & 26\end{vmatrix} = -(26-9) = -17$$

$$A_{31} = Cofactor\ of\ 9 = (-1)^{3+1}\begin{vmatrix}1 & 1\\ 5 & 6\end{vmatrix} = 6-5 = 1$$

$$A_{32} = Cofactor\ of\ 26 = (-1)^{3+2}\begin{vmatrix}1 & 1\\ 3 & 6\end{vmatrix} = -(6-3) = -3$$

$$A_{33} = Cofactor\ of\ 36 = (-1)^{3+3}\begin{vmatrix}1 & 1\\ 3 & 5\end{vmatrix} = 5-3 = 2$$

The matrix formed by cofactors of element of determinant $|A|$ is $\begin{pmatrix}24 & -54 & 33\\ -10 & 27 & -17\\ 1 & -3 & 2\end{pmatrix}$

$$\therefore \text{adj } A = \begin{pmatrix}24 & -10 & 1\\ -54 & 27 & -3\\ 33 & -17 & 2\end{pmatrix}$$

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{3}\begin{pmatrix}24 & -10 & 1\\ -54 & 27 & -3\\ 33 & -17 & 2\end{pmatrix}$$

We Know that $X=A^{-1}B$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 24 & -10 & 1 \\ -54 & 27 & -3 \\ 33 & -17 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ 16 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 24.1 + (-10)4 + 1.16 \\ (-54)1 + 27.4 + (-3)16 \\ 33.1 + (-17)4 + 2.16 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 0 \\ 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

***x* = 0, *y* = 2, *z* = -1.**

SOLUTION BY DETERMINANT (CRAMER'S RULE)

Let the equations be

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3, \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Consider the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad \Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

When $\Delta \neq 0$, the unique solution is given by

$$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta}, z = \frac{\Delta_z}{\Delta}$$

Example

Solve the equations $x + 2y + 5z = 23$, $3x + y + 4z = 26$, $6x + y + 7z = 47$ by determinant method (Cramer’s Rule).

Solution

The equations are

$x + 2y + 5z = 23,$

$3x + y + 4z = 26,$

$6x + y + 7z = 47$

$$\Delta = \begin{vmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 6 & 1 & 7 \end{vmatrix} = -6 \neq 0; \qquad \Delta_x = \begin{vmatrix} 23 & 2 & 5 \\ 26 & 1 & 4 \\ 47 & 1 & 7 \end{vmatrix} = -24;$$

$$\Delta_y = \begin{vmatrix} 1 & 23 & 5 \\ 3 & 26 & 4 \\ 6 & 47 & 7 \end{vmatrix} = -12 \qquad \Delta_z = \begin{vmatrix} 1 & 2 & 23 \\ 3 & 1 & 26 \\ 6 & 1 & 47 \end{vmatrix} = -18$$

By Cramer's rule

$$x = \frac{\Delta_x}{\Delta} = \frac{-24}{-6} = 4$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-12}{-6} = 2$$

$$z = \frac{\Delta_z}{\Delta} = \frac{-18}{-6} = 3$$

$\Rightarrow x = 4, y = 2, z = 3.$

MATRICES

An arrangement of numbers in rows and columns. A matrix of type "(m x n)" is defined as arrangement of (m x n) numbers in 'm' rows & 'n' columns. Usually these numbers are enclosed within square brackets [ ] (or) simple brackets ( ) are denoted by capital letters A, B, C etc.

Example

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 9 & 10 & -1 & 3 \\ 4 & 2 & 8 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 10 & 2 \\ 9 & -1 & 3 \\ 4 & 8 & 5 \end{pmatrix}$$

Here A is of type 3 x 4 & B is of type 4 x 3

Types of matrices

1. Row matrix: It is a matrix containing only one row and several columns .It is also called as row vector.

Example:

$$\begin{bmatrix} 1 & 3 & 7 & 9 & 6 \end{bmatrix}$$

↓

(1 x 5) matrix called row vector.

2. Column matrix: It is a matrix containing only one column. It is also known as column vector.

Example: $\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ (3x1)

3. Square matrix: A matrix is called as square matrix, if the number of rows is equal to number of columns.

$$\begin{pmatrix} 4 & 2 & 4 \\ 1 & 9 & 8 \\ 6 & 5 & 2 \end{pmatrix}$$

Example

The elements a_{11}, a_{22}, a_{33} etc fall along the diagonal & this is called a leading diagonal (or) principal diagonal of the matrix.

4. Trace of the matrix

It is defined as the sum of the elements along the leading diagonal.

In this above matrix the trace of the matrix is

4 + 9 + 2 = 15.

5. Diagonal matrix

It is a square matrix in which all the elements other than in the leading diagonals are zero's.

Eg: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

6.Scalar matrix

It is a diagonal matrix in which all the elements in the leading diagonal are same.

Eg: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

7. Unit matrix or identify matrix

It is a diagonal matrix, in which the elements along the leading diagonal are equal to one. It is denoted by I

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

8. Zero matrix (or) Non-matrix

It is matrix all of whole elements are equal to zero denoted by "O"

Eg: $O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 2 x 3 $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ 2 x 2

9. Triangular matrix

There are two types. 1. Lower Triangular Matrix 2. Upper Triangular Matrix.

Lower Triangular matrix

It is a square matrix in which all the elements above the leading diagonal are zeros.

Eg: $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 6 & 2 \end{pmatrix}$

Upper Triangular matrix

Square matrix in which all the elements below the leading diagonal are zeros

Eg: $\begin{pmatrix} 5 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{pmatrix}$

10. Symmetric matrix

A square m $a_{ij} = a_{ij}$ = $\{a_{ij}\}$ said i = 1 to n ; j = 1 to n said to symmetric, if for all i and j.

Eg: $\begin{pmatrix} 1 & 3 & 4 \\ 3 & 6 & -5 \\ 4 & -5 & 2 \end{pmatrix}$

11. Skew symmetric matrix

A square matrix A = $\{a_{ij}\}$ i = 1 $a_{ij} = -a_{ji}$. to n is called skew symmetric, if for all i & j. Here $a_{ii} = 0$ for all i

Eg: $\begin{pmatrix} 0 & 3 & -4 \\ -3 & 0 & 5 \\ 4 & -5 & 0 \end{pmatrix}$

Algebra of matrices

1. Equality of matrices

Two matrices A & B are equal, if and only if,

(i) Both A & B are of the same type

(ii) Every element of 'B' is the same as the corresponding element of 'A'.

Example

1.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 9 & 10 & -1 & 3 \\ 4 & 2 & 8 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 10 & 2 \\ 9 & -1 & 3 \\ 4 & 8 & 5 \end{pmatrix}$$

Here order of matrix A is not same as order matrix B, the two matrices are not equal.

$A \neq B$

2. Find the value of *a* and *b* given

$$\begin{bmatrix} 4 & 5 \\ a & b \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}$$

Solution:

The given matrices are equal

$\therefore$ $a = 3$, $b = 2$

2. Addition of matrices

Two matrices A & B can be added if and only if,

(i) Both are of the same type.

(ii) The resulting matrix of A & B is also of same type and is obtained by adding the all elements of 'A' to the corresponding elements of 'B'.

Example

1. Find $\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$

Solution

$$\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4+2 & 5+3 \\ 5+2 & 6+1 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 7 & 7 \end{bmatrix}$$

3. Subtraction of the matrices

This can be done, when both the matrices are of same type.

(A-B) is obtained by subtracting the elements of 'A' with corresponding elements of 'B'.

Example

1. Find $\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$

Solution

$$\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4-2 & 5-3 \\ 5-2 & 6-1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$$

4. Multiplication of matrix

They are of two types : 1. By a scalar K B.

2. By a matrix $\longrightarrow$ A x B.

i) Scalar multiplication

To multiply a matrix 'A' by a scalar 'K', then multiply every element of a matrix 'A' by that scalar.

Example

1. Find $2\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix}$

Solution:

$$2\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 10 \\ 10 & 12 \end{bmatrix}$$

ii) Matrix Multiplication

Two matrices A & B can be multiplied to form the matrix product AB, if and only if the number of columns of 1st matrix A is equal to the number of rows of 2nd matrix B. If A is an *(m x p)* and B is an *(p x n)* then the matrix product AB can be formed. AB is a matrix by *(m x n)*.

In this case the matrices A and B are said to be conformable for matrix multiplication.

Example

1. Find $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 6 & 4 \\ 2 & -3 \end{pmatrix}$

Solution

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 6 & 4 \\ 2 & -3 \end{pmatrix} = \begin{bmatrix} 2\times 6+3\times 2 & 2\times 4+3\times -3 \\ 4\times 6+5\times 2 & 4\times 4+5\times -3 \end{bmatrix}$$

$$= \begin{bmatrix} 12+6 & 8+-9 \\ 24+10 & 16+-15 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & -1 \\ 34 & 1 \end{bmatrix}$$

Note: The matrix product AB is different from the matrix product BA.

1. The matrix AB can be formed but not BA

   Eg: A is a (2 x 3) matrix

   B is a (3 x 5) matrix

   AB alone can be formed and it is a (2 x 5) matrix.

2. Even if AB & BA can be formed, they need not be of same type.

   Eg: A is a (2 x 3) matrix

   B is a (3 x 2) matrix

   AB can be formed and is a (2 x 2) matrix

   BA can be formed and is a (3x 3) matrix

3. Even if AB & BA are of the same type, they needn't be equal. Because, they need not be identical.

   Eg: A is a (3 x 3) matrix

   B is a (3 x 3) matrix

   AB is a (3 x 3) matrix

   BA is a (3x 3) matrix

   $AB \neq BA$

The multiplication of any matrix with null matrix the resultant matrix is also a null matrix.

When any matrix (ie.) A is multiplied by unit matrix; the resultant matrix is 'A' itself.

Transpose of a matrix

The Transpose of any matrix ('A') is obtained by interchanging the rows & columns of 'A' and is denoted by A^T. If A is of type *(m x n),* then A^T is of type *(n x m).*

Eg: $A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \\ 4 & 5 \end{pmatrix}$ (3 x 2) $\quad A^T = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 0 & 5 \end{pmatrix}$ (2 x 3)

Properties of transpose of a matrix

1) $(A^T)^T = A$

2) $(AB)^T = B^T A^T$ is known as the reversal Law of Transpose of product of two matrices.

DETERMINANTS

Every square matrix A of order n x n with entries real or complex there exists a number called the determinant of the matrix A denoted by by $|A|$ or det (A). The determinant formed by the elements of A is said to be the determinant of the matrix A..

Consider the 2nd order determinant.

$$|A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Eg:

$$|A| = \begin{vmatrix} 4 & 3 \\ 1 & 0 \end{vmatrix} = 0\text{-}3 = -3$$

Consider the 3rd order determinant,

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

This can be expanded along any row or any column. Usually we expand by the 1st row. On expanding along the 1st row

$$|A| = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Minors

Let A = (a_{ij})be a determinant of order n. The minor of the element a_{ij} is the determinant formed by deleting ith row and jth column in which the element belongs and the cofactor of the element is $A_{ij} = (-i)^{i+j} M_{ij}$ where M is the minor of ith row and jth column .

Example 1 Calculate the determinant of the following matrices.

(*a*) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{pmatrix}$

Solution

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = 1\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1(-2-12) - 2(-3-4) + 3(9-2)$$

$$= -14 + 14 + 21 = 21$$

Singular and Non-Singular Matrices:

Definition

A square matrix 'A' is said to be singular if, $|A| = 0$ and it is called non-singular if $|A| \neq 0$.

Note

Only square matrices have determinants.

Example: Find the solution for the matrix $A = \begin{vmatrix} 2 & 4 & 3 \\ 5 & 1 & 0 \\ 7 & 5 & 3 \end{vmatrix}$

$$\begin{vmatrix} 2 & 4 & 3 \\ 5 & 1 & 0 \\ 7 & 5 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 \\ 5 & 3 \end{vmatrix} - 4\begin{vmatrix} 5 & 0 \\ 7 & 3 \end{vmatrix} + 3\begin{vmatrix} 5 & 1 \\ 7 & 5 \end{vmatrix}$$

$$= 2(3-0) - 4(15-0) + 3(25-7)$$

$$= 6 - 60 + 54 = 0$$

Here $|A| = 0$ .So the given matrix is singular

Properties of determinants

1. **The value of a determinant is unaltered by interchanging its rows and columns.**

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{bmatrix}$ then

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = 1\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1(-2-12) - 2(-3-4) + 3(9-2)$$

$$= -14 + 14 + 21 = 21$$

Let us interchange the rows and columns of A. Thus we get new matrix A_1.

Then

$$\det(A_1)=\begin{vmatrix}1 & 3 & 1\\ 2 & 2 & 3\\ 3 & 4 & -1\end{vmatrix}=1\begin{vmatrix}2 & 3\\ 4 & -1\end{vmatrix}-3\begin{vmatrix}2 & 3\\ 3 & -1\end{vmatrix}+1\begin{vmatrix}2 & 2\\ 3 & 4\end{vmatrix}$$

$$=1(-2-12)-3(-2-9)+1(8-6)$$

$$=-14+33+2=21$$

Hence det (A) = det (A_1).

2. If any two rows (columns) of a determinant are interchanged the determinant changes its sign but its numerical value is unaltered.

Example

Let $A=\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 4\\ 1 & 3 & -1\end{bmatrix}$ then

$$\det(A)=\begin{vmatrix}1 & 2 & 3\\ 3 & 2 & 4\\ 1 & 3 & -1\end{vmatrix}=1\begin{vmatrix}2 & 4\\ 3 & -1\end{vmatrix}-2\begin{vmatrix}3 & 4\\ 1 & -1\end{vmatrix}+3\begin{vmatrix}3 & 2\\ 1 & 3\end{vmatrix}$$

$$=1(-2-12)-2(-3-4)+3(9-2)$$

$$=-14+14+21=21$$

Let A_1 be the matrix obtained from A by interchanging the first and second row. i.e R1 and R2.

Then

$$\det(A_1)=\begin{vmatrix}3 & 2 & 4\\ 1 & 2 & 3\\ 1 & 3 & -1\end{vmatrix}=3\begin{vmatrix}2 & 3\\ 3 & -1\end{vmatrix}-2\begin{vmatrix}1 & 3\\ 1 & -1\end{vmatrix}+4\begin{vmatrix}1 & 2\\ 1 & 3\end{vmatrix}$$

$$=3(-2-9)-2(-1-3)+4(3-2)$$

$$=-33+8+4=-21$$

Hence det (A) = - det (A_1).

3. If two rows (columns) of a determinant are identical then the value of the terminant is zero.

Example

Let $A=\begin{bmatrix}1 & 1 & 3\\ 3 & 3 & 4\\ 1 & 1 & -1\end{bmatrix}$ then

$$\det(A)=\begin{vmatrix}1 & 1 & 3\\ 3 & 3 & 4\\ 1 & 1 & -1\end{vmatrix}=1\begin{vmatrix}3 & 4\\ 1 & -1\end{vmatrix}-1\begin{vmatrix}3 & 4\\ 1 & -1\end{vmatrix}+3\begin{vmatrix}3 & 3\\ 1 & 1\end{vmatrix}$$

$$=1(-3-4)-1(-3-4)+3(3-3)$$

$$=-7+7+0=0$$

Hence $\det(A)=0$

4. If every element in a row (or column) of a determinant is multiplied by a constant "k" then the value of the determinant is multiplied by k.

Example

Let $A=\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 4\\ 1 & 3 & -1\end{bmatrix}$ then

$$\det(A)=\begin{vmatrix}1 & 2 & 3\\ 3 & 2 & 4\\ 1 & 3 & -1\end{vmatrix}=1\begin{vmatrix}2 & 4\\ 3 & -1\end{vmatrix}-2\begin{vmatrix}3 & 4\\ 1 & -1\end{vmatrix}+3\begin{vmatrix}3 & 2\\ 1 & 3\end{vmatrix}$$

$$=1(-2-12)-2(-3-4)+3(9-2)$$

$$=-14+14+21=21$$

Let A_1 be the matrix obtained by multiplying the elements of the first row by 2 (ie. here k =2) then

$$\det(A_1)=\begin{vmatrix}2(1) & 2(2) & 2(3)\\ 3 & 2 & 4\\ 1 & 3 & -1\end{vmatrix}=2\times 1\begin{vmatrix}2 & 4\\ 3 & -1\end{vmatrix}-2\times 2\begin{vmatrix}3 & 4\\ 1 & -1\end{vmatrix}+2\times 3\begin{vmatrix}3 & 2\\ 1 & 3\end{vmatrix}$$

$$=2\left[1(-2-12)-2(-3-4)+3(9-2)\right]$$

$$=2\left[-14+14+21\right]=2(21)$$

Hence det (A) = 2 det (A_1).

5. If every element in any row (column) can be expressed as the sum of two quantities then given determinant can be expressed as the sum of two determinants of the same order with the elements of the remaining rows (columns) of both being the same.

Example

Let $A=\begin{bmatrix}1+2 & 2+4 & 3+6\\ 3 & 2 & 4\\ 1 & 3 & -1\end{bmatrix}$ then

$$\det(A) = \begin{vmatrix} 1+2 & 2+4 & 3+6 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 4 & 6 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix}$$

$$= \det(M1) + \det(M2)$$

$$\det(M1) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = 1\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1(-2-12) - 2(-3-4) + 3(9-2)$$

$$= -14 + 14 + 21 = 21$$

$$\det(M2) = \begin{vmatrix} 2 & 4 & 6 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} - 4\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 6\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 2(-2-12) - 4(-3-4) + 6(9-2)$$

$$= -28 + 28 + 42 = 42$$

$$\det(A) = \det(M1) + \det(M2)$$

$$\textit{Hence} \ \det(A) = 21 + 42 = 63$$

6. A determinant is unaltered when to each element of any row (column) is added to those of several other rows (columns) multiplied respectively by constant factors.

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{bmatrix}$ then

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = 1\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1(-2-12) - 2(-3-4) + 3(9-2)$$

$$= -14 + 14 + 21 = 21$$

Let A_1 be a matrix obtained when the elements C_1 of A are added to those of second column and third column multiplied respectively by constants 2 and 3. Then

$$\det(A_1) = \begin{vmatrix} 1+2(2)+3(3) & 2 & 3 \\ 3+2(2)+3(4) & 2 & 4 \\ 1+2(3)+3(-1) & 3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} + \begin{vmatrix} 2(2) & 2 & 3 \\ 2(2) & 2 & 4 \\ 2(3) & 3 & -1 \end{vmatrix} + \begin{vmatrix} 3(3) & 2 & 3 \\ 3(4) & 2 & 4 \\ 3(-1) & 3 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 3 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 & 3 \\ 4 & 2 & 4 \\ -1 & 3 & -1 \end{vmatrix}$$

$$= 1(-2-12) - 2(-3-4) + 3(9-2) + 2(0) + 3(0)$$

$$= -14 + 14 + 21 = 21$$

PHYSICAL AND ECONOMIC OPTIMUM FOR SINGLE INPUT

Let y = f(x) be a response function. Here x stands for the input that is kgs of fertilizer applied per hectare and y the corresponding output that is kgs of yield per hectare.

We know that the maximum is only when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

This optimum is called physical optimum. We are not considering the profit with respect to the investment, we are interested only in maximizing the profit.

Economic optimum

The optimum which takes into consideration the amount invested and returns is called the economic optimum.

$$\frac{dy}{dx} = \frac{P_x}{P_y}$$

where $P_x \rightarrow$ stands for the per unit price of input that is price of fertilizer per kgs.

$P_y \rightarrow$ stands for the per unit price of output that is price of yield per kgs.

Problem

The response function of paddy is y = 1400 + 14.34x -0.05 x^2 where x represents kgs of nitrogen/hectare and y represents yield in kgs/hectare. 1 kg of paddy is Rs. 2 and 1 kg of nitrogen is Rs. 5. Find the physical and economic optimum. Also find the corresponding yield.

Solution

y = 1400 + 14.34x -0.05 x^2

$$\frac{dy}{dx} = 14.34 - 0.1x$$

$\frac{d^2y}{dx^2} = -0.1 =$ negative value

ie. $\frac{d^2y}{dx^2} < 0$

Therefore the given function has a maximum point.

Physical Optimum

$$\frac{dy}{dx} = 0$$

i.e 14.34-0.1x = 0

-0.1 x = -14.34

$x = \frac{14.34}{0.1} = 143.4$ kgs/hectare

therefore the physical optimum level of nitrogen is 143.4 kgs/hectare.

Therefore the maximum yield is

$Y = 1400 + 14.34(143.4) - 0.05(143.4)^2$

$= 2428.178$ kgs/ hectare.

Economic optimum

$$\frac{dy}{dx} = \frac{P_x}{P_y}$$

Given

Price of nitrogen per kg = $P_x = 5$

Price of yield per kg = $P_y = 2$

Therefore $\frac{dy}{dx} =$ $14.34 - 0.1x = \frac{5}{2}$

$28.68 - 0.2x = 5$

$-0.2x = 5 - 28.68$

$x = \frac{23.68}{0.2} = 118.4$ kgs/hectare

therefore the economic optimum level of nitrogen is 118.4 kgs/hectare.

Therefore the maximum yield is

$Y = 1400 + 14.34(118.4) - 0.05(118.4)^2$

$= 2396.928$ kgs/ hectare.

Maxima and Minima of several variables with constraints and without constraints

Consider the function of several variables

$$y = f(x_1, x_2 \ldots\ldots\ldots x_n)$$

where $x_1, x_2 \ldots\ldots\ldots\ldots x_n$ are n independent variables and y is the dependent variable.

Working Rule

Step 1: Find all the first order partial derivatives of y with respect to $x_1, x_2, x_3 \ldots\ldots x_n$..

(ie) $\frac{\partial y}{\partial x_1} = f_1$

$\frac{\partial y}{\partial x_2} = f_2$

$$\frac{\partial y}{\partial x_3} = f_3$$

.

.

.

.

$$\frac{\partial y}{\partial x_n} = fn$$

Step 2

Find all the second order partial derivatives of y with respect to x_1, x_2, x_3 ….x_n and they are given as follows.

$$\frac{\partial^2 y}{\partial x_1^{\;2}} = f_{11}$$

$$\frac{\partial^2 y}{\partial x_2\ \partial x_1} = f_{21}$$

$$\frac{\partial^2 y}{\partial x_2^{\;2}} = f_{22}$$

$$\frac{\partial^2 y}{\partial x_1\ \partial x_2} = f_{12}$$

$$\frac{\partial^2 y}{\partial x_3^{\;2}} = f_{33}$$

$$\frac{\partial^2 y}{\partial x_3\ \partial x_1} = f_{31}$$

$$\frac{\partial^2 y}{\partial x_1\ \partial x_3} = f_{13} \quad \text{and so on}$$

Step: 3

Construct an Hessian matrix which is formed by taking all the second order partial derivatives is given by

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \cdots\cdots\cdots f_{1n} \\ f_{21} & f_{22} & f_{23} \cdots\cdots\cdots f_{2n} \\ . \\ . \\ . \\ . \\ . \\ f_{n1} & f_{n2} & f_{n3} \cdots\cdots\cdots f_{nn} \end{bmatrix}$$

H is a symmetric matrix.

Step: 4

Consider the following minors of order 1, 2, 3 ……….

$$|H_1| = |f_{11}| = f_{11}$$

$$|H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

$$|H_3| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

.
.
.
.
.

$$|H_n| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \cdots\cdots\cdots f_{1n} \\ f_{21} & f_{22} & f_{23} \cdots\cdots\cdots f_{2n} \\ . & . & . \quad\quad\quad . \\ . & . & . \quad\quad\quad . \\ . & . & . \quad\quad\quad . \\ . & . & . \quad\quad\quad . \\ f_{n1} & f_{n2} & f_{n3} \cdots\cdots\cdots f_{nn} \end{vmatrix}$$

Steps: 5

The necessary condition for finding the maximum or minimum

Equate the first order derivative to zero (i.e) $f_1 = f_2 = \ldots\ldots f_n = 0$ and find the value of $x_1, x_2, \ldots\ldots x_n$.

Steps: 6

Substitute the values $x_1, x_2 \ldots\ldots x_n$ in the Hessian matrix. Find the values of $|H_1|, |H_2|, |H_3| \ldots\ldots\ldots |H_n|$

$$\text{If} \quad \begin{array}{l} |H_1| < 0 \\ |H_2| > 0 \\ |H_3| < 0 \\ |H_4| > 0 \ldots\ldots\ldots \textit{ and so on.} \end{array}$$

Then the function is **maximum** at $x_1, x_2 \ldots\ldots x_n$.

If $|H_1| > 0, \quad |H_2| > 0, \quad |H_3| > 0 \ldots\ldots\ldots$ then the function is **minimum** at $x_1, x_2 \ldots\ldots x_n$.

Steps: 7

Conditions	Maximum	Minimum
First	$f_1 = f_2 = f_3 = f_n = 0$	$f_1 = 0, f_2 = 0 \ldots\ldots f_n = 0$
Second	$\|H_1\| < 0$ $\|H_2\| > 0$ $\|H_3\| < 0$ $\ldots\ldots\ldots$	$\|H_1\| > 0$ $\|H_2\| > 0$ $\|H_3\| > 0$ . . .

Note :

If the second order conditions are not satisfied then they are called **saddle point.**

Problem

Find the maxima (or) minima if any of the following function.

$$y = \frac{4}{3}x_1^3 + x_2^2 - 4x_1 + 8x_2 \qquad \text{\_\_\_\_\_\_\_\_(1)}$$

Solution

Step 1: The first order partial derivatives are

$$f_1 = \frac{\partial y}{\partial x_1} = 4x_1^{\,2} - 4$$

$$f_2 = \frac{\partial y}{\partial x_2} = 2x_2 + 8$$

Step 2: The second order partial derivatives are

$$f_{11} = \frac{\partial^2 y}{\partial x_1^{\,2}} = 8x_1$$

$$f_{21} = \frac{\partial^2 y}{\partial x_2\, \partial x_1} = 0$$

$$f_{22} = \frac{\partial^2 y}{\partial x_2^{\,2}} = 2$$

$$f_{12} = \frac{\partial^2 y}{\partial x_1\, \partial x_2} = 0$$

Step 3: The Hessian matrix is $H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$

$$H = \begin{bmatrix} 8x_1 & 0 \\ 0 & 2 \end{bmatrix}$$

4. Equate $f_1, f_2 = 0$

$f_1 \Rightarrow 4x_1^{2} - 4 = 0$

$x_1^2 = 1$

$x_1 = \pm 1$

$x_1 = 1,\ x_1 = -1$

$f_2 \Rightarrow 2x_2 + 8 = 0$

$2x_2 = -8$

$x_2 = -4$

The stationary points are (1,- 4) & (-1, - 4)

At the point (1, - 4) the Hessian matrix will be

$$H = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$$

$|H_1| = |8| > 0$

$$|H_2| = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} = 16 > 0$$

Since the determinant H_1 and H_2 are positive the function is minimum at (1, - 4).
The minimum value at $x_1 = 1$ & $x_2 = -4$ is obtained by substituting the values in (1)

$$y = \frac{4}{3}(1)^3 + (-4)^2 - 4(1) + 8(-4)$$

$$y = \frac{4}{3} + 16 - 4 - 32$$

$$y = \frac{4}{3} - 20$$

$$y = \frac{4-60}{3} = \frac{-56}{3}$$

The minimum value is $\frac{-56}{3}$

<u>At the point (-1, - 4)</u>

$$H = \begin{vmatrix} -8 & 0 \\ 0 & 2 \end{vmatrix}$$

$|H_1| = |-8| = -8 < 0$

$|H_2| = -16 < 0$

Both the conditions are not satisfied. Hence the point (-1, - 4) gives a saddle point.

Economic Optimum

For finding the Economic Optimum we equate the first order derivative $f_1 . f_2 .... f_n$ to the inverse ratio of the unit prices.

(ie) $f_1 = \frac{\partial y}{\partial x_1} = \frac{p_{x_1}}{p_y}$

$f_2 = \frac{\partial y}{\partial x_2} = \frac{p_{x_2}}{p_y}$ ....

$f_n = \frac{\partial y}{\partial x_n} = \frac{p_{x_n}}{p_y}$

where Px_1, Px_2,... Px_n and Py are the unit prices of $x_1, x_2 \ldots x_n$ and y. These are the first order condition.

The economic optimum & the physical optimum differ only in the first order conditions. The other procedures are the same.

Maxima & Minima of several variables under certain condition with constraints.

Consider the response function

$y = f(x_1, x_2 \ldots x_n)$ subject to the constraint $\phi(x_1, x_2 \ldots x_n) = 0$

The objective function is Z= f(x1, x2, …xn) + λ[ϕ(x1, x2, …xn)]

where λ is called the Lagrange's multiplies.

The partial derivatives are

$$\frac{\partial z}{\partial x_i} = fi \qquad \text{for } i = 1, 2 \ldots n.$$

$$\frac{\partial^2 z}{\partial x_i \partial x_1} = f_{ij} \qquad i, j = 1., 2 \ldots n.$$

$$\frac{\partial \phi}{\partial x_i} = \phi i \qquad i = 1, 2 \ldots n.$$

Now form the Bordered Hessian Matrix as follows.

$$\text{Bordered Hessian} \quad \overline{H} = \begin{bmatrix} 0 & \phi_1 & \phi_2 \ldots\ldots\ldots\ldots\phi \\ \phi_1 & f_{11} & f_{12} \ldots\ldots\ldots\ldots f_{1n} \\ \phi_2 & f_{21} & f_{22} \ldots\ldots\ldots\ldots f_{2n} \\ . & . & \\ . & & \\ \phi_2 & f_{n1} & f_{n2} \ldots\ldots\ldots\ldots f_{nn} \end{bmatrix}$$

[Since this extra row & column is on the border of the matrix $\begin{bmatrix} f_{11} & f_{12} \ldots\ldots\ldots f_{1n} \\ f_{21} & f_{22} \ldots\ldots\ldots f_{2n} \\ f_{n1} & f_{n2} \ldots\ldots\ldots f_{nn} \end{bmatrix}$.So we call it as Bordered Hessian matrix and it is denoted by $\overline{H}$]

Here minor as are

$$\left[\overline{H}_1\right] = \begin{vmatrix} 0 & \phi_1 \\ \phi_1 & f_{1i} \end{vmatrix}, \quad \left|\overline{H}_2\right| = \begin{vmatrix} 0 & \phi_1 & \phi_2 \\ \phi_1 & f_{11} & f_{12} \\ \phi_2 & f_{21} & f_{22} \end{vmatrix}$$

$$|\overline{H}_3| = \begin{vmatrix} 0 & \phi_1 & \phi_2 & \phi_3 \\ \phi_1 & f_{11} & f_{12} & f_{13} \\ \phi_2 & f_{21} & f_{22} & f_{23} \\ \phi_3 & f_{31} & f_{32} & f_{33} \end{vmatrix}$$ and so on.

Problem

Conditions	Maxima	Minima
First Order	$f_1=f_2= f_3 = \ldots .f_n =0$	$f_{1=}\ f_{2=} f_3 \ldots\ldots f_n = 0$
Second Order	$\lvert\overline{H}_2\rvert > 0 \lvert\overline{H}_3\rvert < 0,\ \lvert\overline{H}_4\rvert > 0 \ldots.$	$\lvert\overline{H}_2\rvert < 0\ ,\ \lvert\overline{H}_3\rvert < 0,\ \lvert\overline{H}_4\rvert < 0\ \ldots.$

Consider a consumer with a simple utility function U = f(x, y) = $4xy - y^2$. If this consumer can at most spend only Rs. 6/- on two goods x and y and if the current prices are Rs. 2/- per unit of x and Rs.1/- per unit of y. Maximize the function.

Definition

Model

A mathematical model is a representation of a phenomena by means of mathematical equations. If the phenomena is growth, the corresponding model is called a growth model. Here we are going to study the following 3 models.

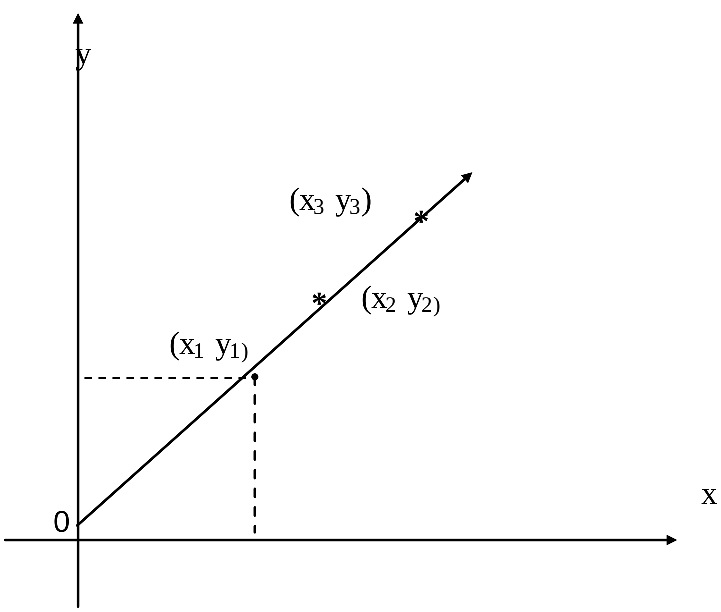
1. linear model
2. Exponential model
3. Power model

1. Linear model

The general form of a linear model is y = a+bx. Here both the variables x and y are of degree 1.

To fit a linear model of the form y=a+bx to the given data.

Here a and b are the parameters (or) constants of the model. Let (x_1, y_1) (x_2, y_2)………… (x_n, y_n) be n pairs of observations. By plotting these points on an ordinary graph sheet, we get a collection of dots which is called a scatter diagram.

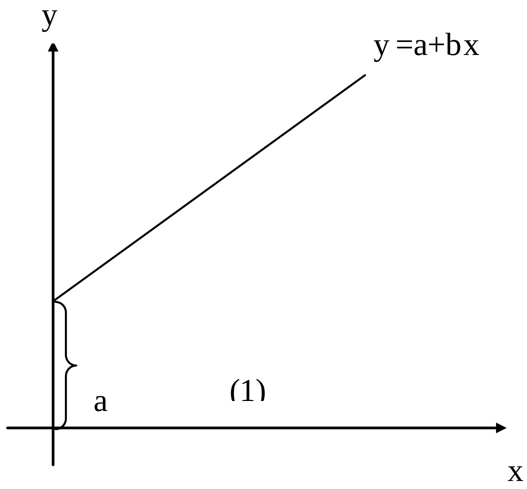


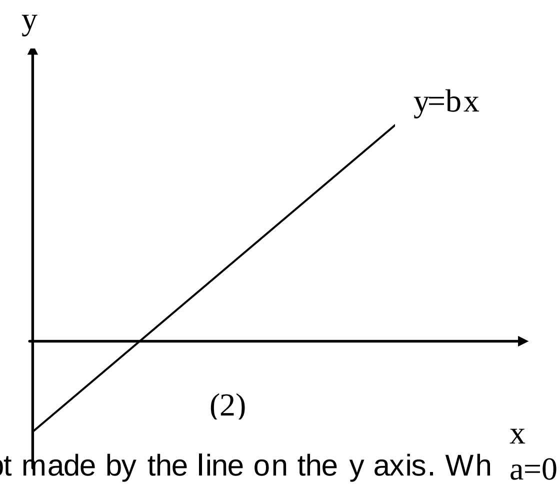
There are two types of linear models

(i) y = a+bx (with constant term)

(ii) y = bx (without constant term)

The graphs of the above models are given below :





'a' stands for the constant term which is the intercept made by the line on the y axis. Wh =0, y =a ie 'a' is the intercept, 'b' stands for the slope of the line .

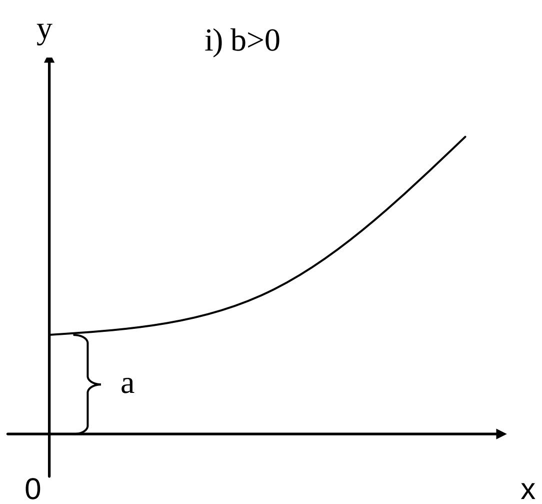
Eg:1. The table below gives the DMP(kgs) of a particular crop taken at different stages; fit a linear growth model of the form w=a+bt, and find the value of a and b from the graph.

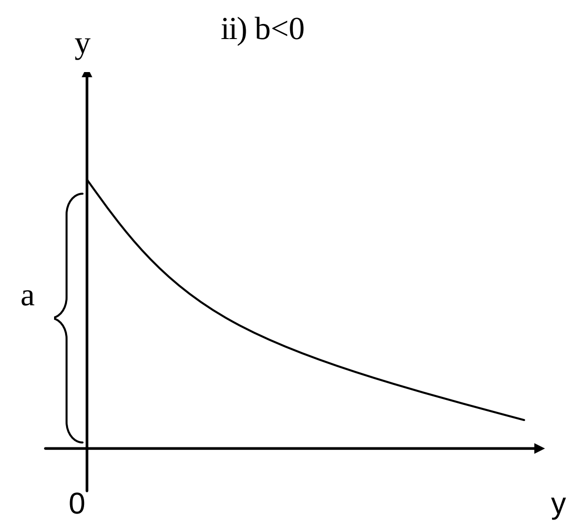
t (in days) ;	0	5	10	20	25
DMP w: (kg/ha)	2	5	8	14	17

2. Exponential model

This model is of the form $y = ae^{bx}$ where a and b are constants to be determined

The graph of an exponential model is given below.





Note:

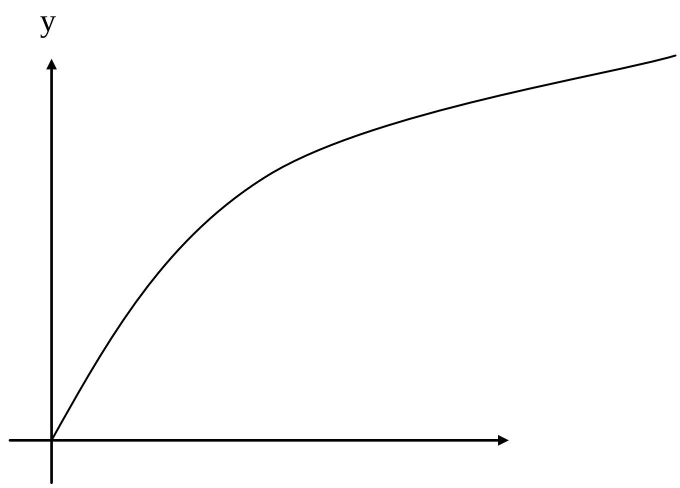
The above model is also known as a semilog model. When the values of x and y are plotted on a semilog graph sheet we will get a straight line. On the other hand if we plot the points xand y on an ordinary graph sheet we will get an exponential curve.

Eg: 2. Fit an exponential model to the following data.

x in days	5	15	25	35	45
y in mg per plant	0.05	0.4	2.97	21.93	162.06

Power Model

The most general form of the power model is $y = ax^b$



o x

Example: Fit the power function for the following data

x	0	1	2	3
y	0	2	16	54

Crop Response models

The most commonly used crop response models are

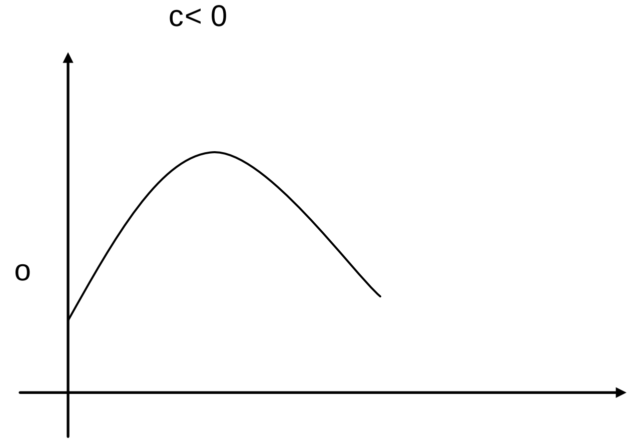
i) Quadratic model

ii) Square root model

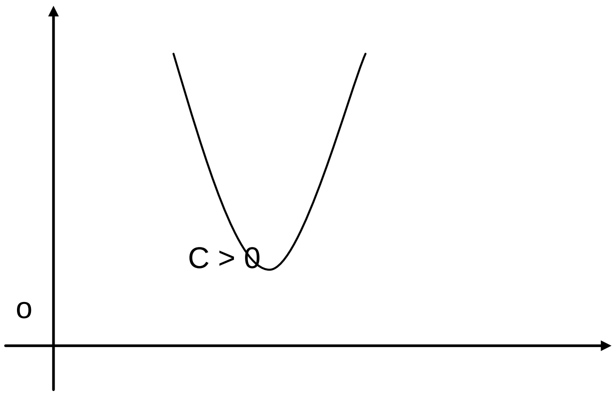
Quadratic model

The general form of quadratic model is $y = a + b x + c x^2$

When c< 0 the curve attains maximum at its peak.



When c > 0 the curve attains minimum at its peak.



The parabolic curve bends very sharply at the maximum or minimum points.

Example

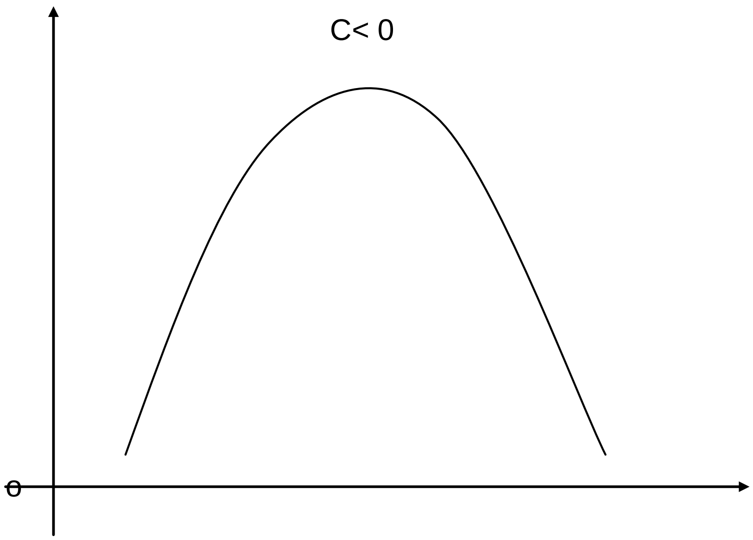
Draw a curve of the form $y = a + b x + c x^2$ using the following values of x and y

x	0	1	2	4	5	6
y	3	4	3	-5	-12	-21

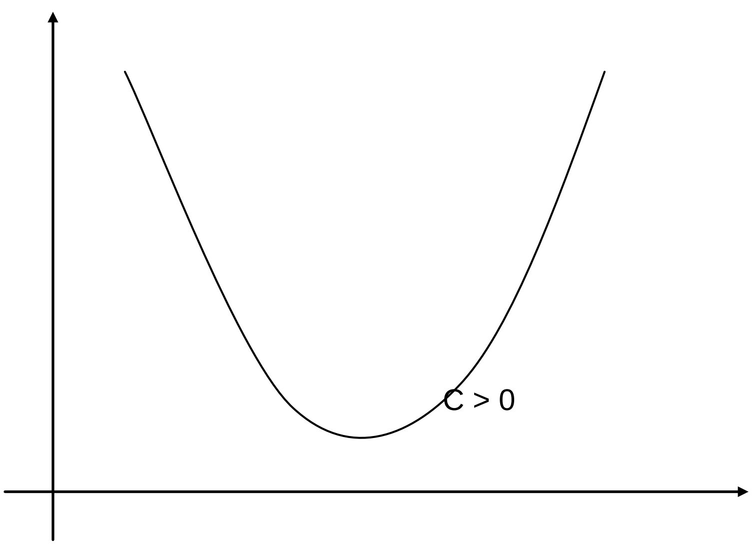
Square root model

The standard form of the square root model is $y = a + b\sqrt{x} + cx$

When c is negative the curve attains maximum



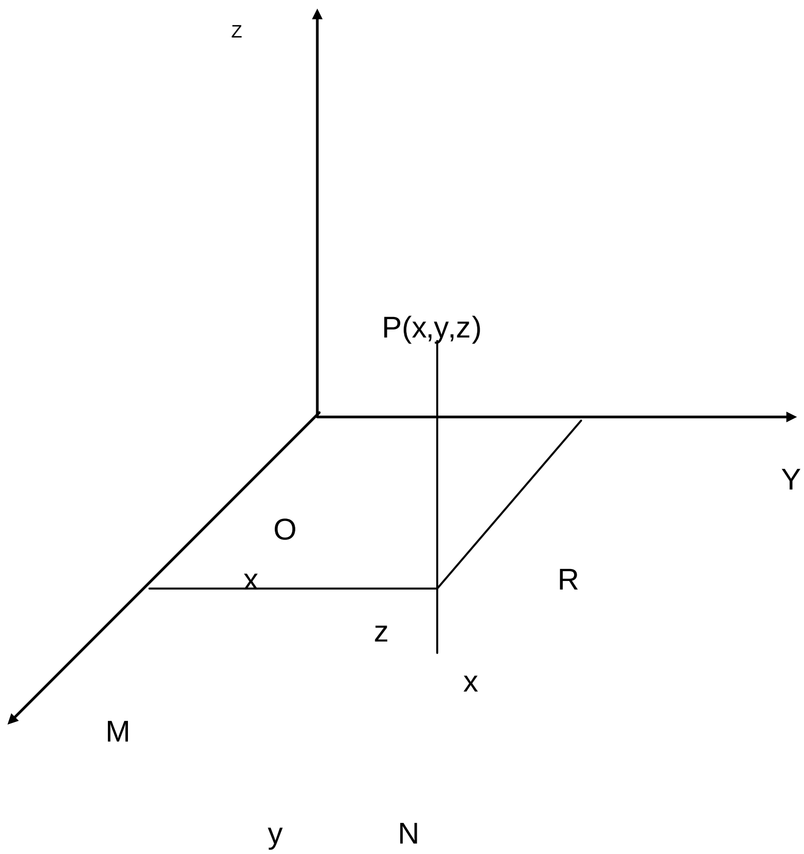
The curve attains minimum when c is positive.



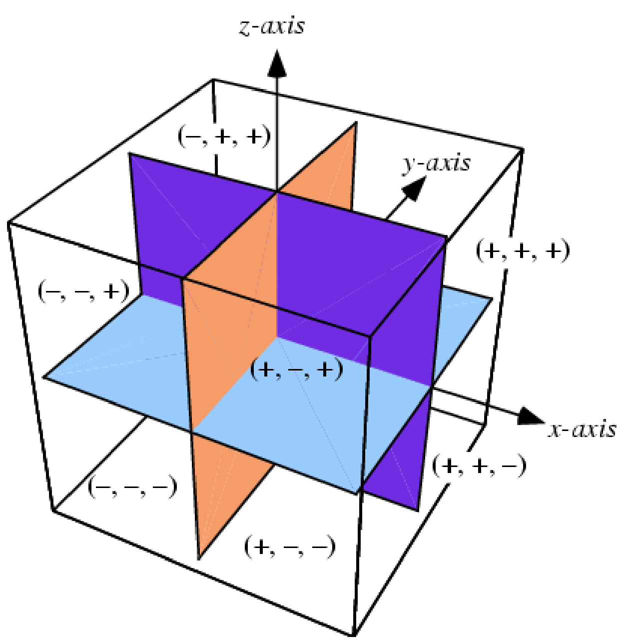
At the extreme points the curve bends at slower rate

Three dimensional Analytical geometry

Let OX ,OY & OZ be mutually perpendicular straight lines meeting at a point O. The extension of these lines OX^1, OY^1 and OZ^1 divide the space at O into octants(eight). Here mutually perpendicular lines are called X, Y and Z co-ordinates axes and O is the origin. The point P (x, y, z) lies in space where x, y and z are called x, y and z coordinates respectively.



where NR = x coordinate, MN = y coordinate and PN = z coordinate



Distance between two points

The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

dist AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

In particular the distance between the origin O (0,0,0) and a point P(x,y,z) is

OP = $\sqrt{x^2 + y^2 + z^2}$

The internal and External section

Suppose $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points in three dimensions.

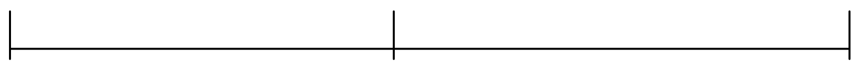
$P(x_1,y_1,z_1)$ A(x, y, z) $Q(x_2,y_2,z_2)$

The point A(x, y, z) that divides distance PQ internally in the ratio $m_1:m_2$ is given by

$$A = \left[\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right]$$

Similarly

$P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points in three dimensions.



$P(x_1,y_1,z_1)$ $Q(x_2,y_2,z_2)$ A(x, y, z)

The point A(x, y, z) that divides distance PQ externally in the ratio $m_1:m_2$ is given by

$$A = \left[\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \frac{m_1z_2 - m_2z_1}{m_1 - m_2}\right]$$

If A(x, y, z) is the midpoint then the ratio is 1:1

$$A = \left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right]$$

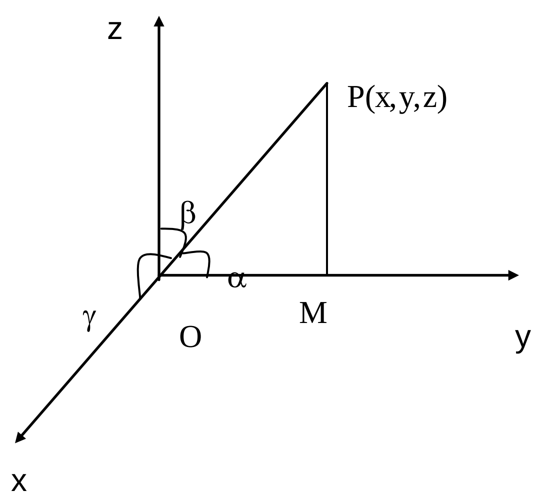
Problem

Find the distance between the points P(1,2-1) & Q(3,2,1)

PQ= $\sqrt{(3-1)^2 + (2-2)^2 + (1+1)^2} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

Direction Cosines

Let P(x, y, z) be any point and OP = r. Let α, β, γ be the angle made by line OP with OX, OY & OZ. Then α, β, γ are called the direction angles of the line OP. cos α, cos β, cos γ are called the direction cosines (or dc's) of the line OP and are denoted by the symbols l, m ,n.



Result

By projecting OP on OY, PM is perpendicular to y axis and the $\angle POM = \beta$ also OM = y

$$\therefore \cos\beta = \frac{y}{r}$$

Similarly, $\cos\alpha = \frac{x}{r}$

$$\cos\gamma = \frac{z}{r}$$

(i.e) $l = \frac{x}{r}$, $m = \frac{y}{r}$, $n = \frac{z}{r}$

$$\therefore l^2 + m^2 + n^2 = \frac{x^2+y^2+z^2}{r^2}$$

$(\because r = \sqrt{x^2+y^2+z^2} \Rightarrow$ Distance from the origin)

$$\therefore l^2 + m^2 + n^2 = \frac{x^2+y^2+z^2}{x^2+y^2+z^2} = 1$$

$l^2 + m^2 + n^2 = 1$

(or) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Note

The direction cosines of the x axis are (1,0,0)

The direction cosines of the y axis are (0,1,0)

The direction cosines of the z axis are (0,0,1)

Direction ratios

Any quantities, which are proportional to the direction cosines of a line, are called direction ratios of that line. Direction ratios are denoted by a, b, c.

If l, m, n are direction cosines an a, b, c are direction ratios then

$a \propto l$, $b \propto m$, $c \propto n$

(ie) $a = kl$, $b = km$, $c = kn$

(ie) $\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = k$ (Constant)

(or) $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{1}{k}$ (Constant)

To find direction cosines if direction ratios are given

If a, b, c are the direction ratios then direction cosines are

$$\left.\begin{aligned} \frac{l}{a} = \frac{1}{k} \Rightarrow \quad & l = \frac{a}{k} \\ \text{similarly} \quad & m = \frac{b}{k} \\ & n = \frac{c}{k} \end{aligned}\right\} \quad (1)$$

$l^2+m^2+n^2 = \frac{1}{k^2}(a^2+b^2+c^2)$

(ie) $1 = \frac{1}{k^2}(a^2+b^2+c^2)$

$$\Rightarrow k^2 = a^2+b^2+c^2$$

Taking square root on both sides

K = $\sqrt{a^2+b^2+c^2}$

$$\therefore l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

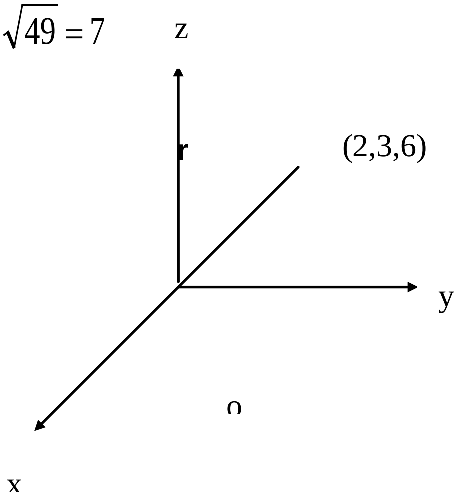
Problem

1. Find the direction cosines of the line joining the point (2,3,6) & the origin.

Solution

By the distance formula

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{2^2+3^2+6^2} = \sqrt{4+9+36} = \sqrt{49} = 7$$



Direction Cosines are

***l* = cos** $\propto$ **=** $\frac{x}{r} = \frac{2}{7}$

***m* = cos** β **=** $\frac{y}{r} = \frac{3}{7}$

***n* = cos** γ **=** $\frac{z}{r} = \frac{6}{7}$

2. Direction ratios of a line are 3,4,12. Find direction cosines

Solution

Direction ratios are 3,4,12

(ie) a = 3, b = 4, c = 12

Direction cosines are

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{3}{\sqrt{3^2+4^2+12^2}} = \frac{3}{\sqrt{169}} = \frac{3}{13}$$

$$m = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{4}{\sqrt{3^2+4^2+12^2}} = \frac{4}{\sqrt{169}} = \frac{4}{13}$$

$$n = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{12}{\sqrt{3^2+4^2+12^2}} = \frac{12}{\sqrt{169}} = \frac{12}{13}$$

Note

1) The direction ratios of the line joining the two points A(x_1, y_1, z_1) & B (x_2, y_2, z_2) are ($x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$)

2) The direction cosines of the line joining two points A (x_1, y_1, z_1) & B (x_2, y_2, z_2) are $\frac{x_2 - x_1}{r}, \frac{y_2 - y_1}{r}, \frac{z_2 - z_1}{r}$

r = distance between AB.

CHAIN RULE DIFFERENTIATION

If y is a function of u ie y = f(u) and u is a function of x ie u = g(x) then y is related to x through the intermediate function u ie y =f(g(x))

$\therefore$ y is differentiable with respect to x

Furthermore, let y=f(g(x)) and u=g(x), then

$$\frac{dy}{dx} = \frac{dy}{du}\ \frac{du}{dx}$$

There are a number of related results that also go under the name of "chain rules." For example, if y=f(u) u=g(v), and v=h(x),

then $$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}$$

Problem

Differentiate the following with respect to x

1. y = $(3x^2+4)^3$
2. y = $e^{x^{-2}}$

Marginal Analysis

Let us assume that the total cost C is represented as a function total output q. (i.e) C= f(q).

Then marginal cost is denoted by MC= $\frac{dc}{dq}$

The average cost = $\frac{TC}{Q}$

Similarly if U = u(x) is the utility function of the commodity x then

the marginal utility MU = $\frac{dU}{dx}$

The total revenue function TR is the product of quantity demanded Q and the price P per unit of that commodity then TR = Q.P = f(Q)

Then the marginal revenue denoted by MR is given by $\frac{dR}{dQ}$

The average revenue = $\frac{TR}{Q}$

Problem

1. If the total cost function is $C = Q^3 - 3Q^2 + 15Q$. Find Marginal cost and average cost.

Solution:

MC = $\frac{dc}{dq}$

AC = $\frac{TC}{Q}$

2. The demand function for a commodity is P= (a - bQ). Find marginal revenue.

(the demand function is generally known as Average revenue function). Total revenue TR = P.Q = Q. (a - bQ) and marginal revenue MR= $\frac{d\left(aQ - bQ^2\right)}{dq}$

Growth rate and relative growth rate

The growth of the plant is usually measured in terms of dry mater production and as denoted by W. Growth is a function of time t and is denoted by W=g(t) it is called a growth function. Here t is the independent variable and w is the dependent variable.

The derivative $\frac{dw}{dt}$ is the growth rate (or) the absolute growth rate gr= $\frac{dw}{dt}$. GR = $\frac{dw}{dt}$

The relative growth rate i.e defined as the absolute growth rate divided by the total dry matter production and is denoted by RGR.

i.e RGR = $\frac{1}{w} . \frac{dw}{dt} = \frac{absolute\,growth\,rate}{total\,dry\,matter\,production}$

Problem

1. If $G = at^2 + b \sin t + 5$ is the growth function function the growth rate and relative growth rate.

   GR = $\frac{dG}{dt}$

   RGR = $\frac{1}{G} . \frac{dG}{dt}$

Implicit Functions

If the variables x and y are related with each other such that f (x, y) = 0 then it is called Implicit function. A function is said to be **explicit** when one variable can be expressed completely in terms of the other variable.

For example, $y = x^3 + 2x^2 + 3x + 1$ is an Explicit function

$xy^2 + 2y + x = 0$ is an implicit function

Problem

For example, the implicit equation xy=1 can be solved by differentiating implicitly gives

$$\frac{d(xy)}{dx} = \frac{d(1)}{dx}$$

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}.$$

Implicit differentiation is especially useful when y'(x)is needed, but it is difficult or inconvenient to solve for y in terms of x.

Example: *Differentiate the following function with respect to x* $x^3y^6 + \mathbf{e}^{1-x} - \cos(5y) = y^2$

Solution

So, just differentiate as normal and tack on an appropriate derivative at each step. Note as well that the first term will be a product rule.

$$3x^2x'y^6 + 6x^3y^5y' - x'\mathbf{e}^{1-x} + 5y'\sin(5y) = 2yy'$$

Example: Find y' for the following function.

$$x^2 + y^2 = 9$$

Solution

In this example we really are going to need to do implicit differentiation of *x* and write *y* as *y(x)*.

$$\frac{d}{dx}\left(x^2 + [y(x)]^2\right) = \frac{d}{dx}(9)$$

$$2x + 2[y(x)]^1 y'(x) = 0$$

Notice that when we differentiated the *y* term we used the chain rule.

Example:

Find y' for the following. $x^3y^5 + 3x = 8y^3 + 1$

Solutio

First differentiate both sides with respect to *x* and notice that the first time on left side will be a product rule.

$$3x^2y^5 + 5x^3y^4y' + 3 = 24y^2y'$$

Remember that very time we differentiate a *y* we also multiply that term by $y'y'$ since we are just using the chain rule. Now solve for the derivative.

$$3x^2y^5+3=24y^2y'-5x^3y^4y'$$
$$3x^2y^5+3=\left(24y^2-5x^3y^4\right)y'$$
$$y'=\frac{3x^2y^5+3}{24y^2-5x^3y^4}$$

The algebra in these can be quite messy so be careful with that.

Example

Find y' for the following $x^2\tan(y)+y^{10}\sec(x)=2x$

Here we've got two product rules to deal with this time.

$$2x\tan(y)+x^2\sec^2(y)y'+10y^9y'\sec(x)+y^{10}\sec(x)\tan(x)=2$$

Notice the derivative tacked onto the secant. We differentiated a *y* to get to that point and so we needed to tack a derivative on.

Now, solve for the derivative.

$$\left(x^2\sec^2(y)+10y^9\sec(x)\right)y'=2-y^{10}\sec(x)\tan(x)-2x\tan(y)$$
$$y'=\frac{2-y^{10}\sec(x)\tan(x)-2x\tan(y)}{x^2\sec^2(y)+10y^9\sec(x)}$$

Logarithmic Differentiation

For some problems, first by taking logarithms and then differentiating, it is easier to find $\frac{dy}{dx}$. Such process is called Logarithmic differentiation.

(i) If the function appears as a product of many simple functions then by taking logarithm so that the product is converted into a sum. It is now easier to differentiate them.

(ii) If the variable x occurs in the exponent then by taking logarithm it is reduced to a familiar form to differentiate.

Example Differentiate the function.

$$y=\frac{x^5}{(1-10x)\sqrt{x^2+2}}$$

Solution Differentiating this function could be done with a product rule and a quotient rule. We can simplify things somewhat by taking logarithms of both sides.

$$\ln y = \ln\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right)$$

$$\ln y = \ln\left(x^5\right) - \ln\left((1-10x)\sqrt{x^2+2}\right)$$

$$\ln y = \ln\left(x^5\right) - \ln(1-10x) - \ln\left(\sqrt{x^2+2}\right)$$

$$\frac{y'}{y} = \frac{5x^4}{x^5} - \frac{-10}{1-10x} - \frac{\frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}(2x)}{\left(x^2+1\right)^{\frac{1}{2}}}$$

$$\frac{y'}{y} = \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+1}$$

Example Differentiate $y = x^x$

Solution

First take the logarithm of both sides as we did in the first example and use the logarithm properties to simplify things a little.

$$\ln y = \ln x^x$$
$$\ln y = x \ln x$$

Differentiate both sides using implicit differentiation.

$$\frac{y'}{y} = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$$

As with the first example multiply by *y* and substitute back in for *y*.

$$y' = y(1+\ln x)$$
$$= x^x(1+\ln x)$$

PARAMETRIC FUNCTIONS

Sometimes variables x and y are expressed in terms of a third variable called **parameter.** We find $\frac{dy}{dx}$ without eliminating the third variable.

Let x = f(t) and y = g(t) then

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} = \frac{dy/dt}{dx/dt}$$

Problem

1. Find for the parametric function x =a cos θ , y = b sin θ

Solution

$$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{b\cos\theta}{-a\sin\theta}$$

$$= -\frac{b}{a}\cot\theta$$

Inference of the differentiation

Let y = f(x) be a given function then the first order derivative is $\frac{dy}{dx}$.

The geometrical meaning of the first order derivative is that it represents the slope of the curve y = f(x) at x.

The physical meaning of the first order derivative is that it represents the rate of change of y with respect to x.

PROBLEMS ON HIGHER ORDER DIFFERENTIATION

The rate of change of y with respect x is denoted by $\frac{dy}{dx}$ and called as the first order derivative of function y with respect to x.

The first order derivative of y with respect to x is again a function of x, which again be differentiated with respect to x and it is called second order derivative of y = f(x) and is denoted by $\frac{d^2y}{dx^2}$ which is equal to $\frac{d}{dx}\left(\frac{dy}{dx}\right)$

In the similar way higher order differentiation can be defined. Ie. The nth order derivative of y=f(x) can be obtained by differentiating n-1th derivative of y=f(x)

$$\frac{d^n y}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1} y}{dx^{n-1}}\right)$$ where n= 2,3,4,5….

Problem

Find the first , second and third derivative of

1. y = e^{ax+b}
2. y = log(a-bx)
3. y = sin (ax+b)

Partial Differentiation

So far we considered the function of a single variable y = f(x) where x is the only independent variable. When the number of independent variable exceeds one then we call it as the function of several variables.

Example

z = f(x,y) is the function of two variables x and y , where x and y are independent variables.

U=f(x,y,z) is the function of three variables x,y and z , where x, y and z are independent variables.

In all these functions there will be only one dependent variable.

Consider a function z = f(x,y). The partial derivative of z with respect to x denoted by $\frac{\partial z}{\partial x}$ and is obtained by differentiating z with respect to x keeping y as a constant. Similarly the partial derivative of z with respect to y denoted by $\frac{\partial z}{\partial y}$ and is obtained by differentiating z with respect to y keeping x as a constant.

Problem

1. Differentiate U = log (ax+by+cz) partially with respect to x, y & z

We can also find higher order partial derivatives for the function z = f(x,y) as follows

(i) The second order partial derivative of z with respect to x denoted as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$ is obtained by partially differentiating $\frac{\partial z}{\partial x}$ with respect to x. this is also known as direct second order partial derivative of z with respect to x.

(ii)The second order partial derivative of z with respect to y denoted as $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}$ is obtained by partially differentiating $\frac{\partial z}{\partial y}$ with respect to y this is also known as direct second order partial derivative of z with respect to y

(iii) The second order partial derivative of z with respect to x and then y denoted as $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial y \partial x}$ is obtained by partially differentiating $\frac{\partial z}{\partial x}$ with respect to y. this is also known as mixed second order partial derivative of z with respect to x and then y

iv) The second order partial derivative of z with respect to y and then x denoted as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x \partial y}$ is obtained by partially differentiating $\frac{\partial z}{\partial y}$ with respect to x. this is also known as mixed second order partial derivative of z with respect to y and then x.

In similar way higher order partial derivatives can be found.

Problem

Find all possible first and second order partial derivatives of

1) z = sin(ax +by)

2) u = xy + yz + zx

Homogeneous Function

A function in which each term has the same degree is called a homogeneous function.

Example

1) $x^2 - 2xy + y^2 = 0 \rightarrow$ homogeneous function of degree 2.
2) $3x + 4y = 0 \rightarrow$ homogeneous function of degree 1.
3) $x^3 + 3x^2y + xy^2 - y^3 = 0 \rightarrow$ homogeneous function of degree 3.

To find the degree of a homogeneous function we proceed as follows.

Consider the function f(x,y) replace x by tx and y by ty if $f(tx, ty) = t^n f(x, y)$ then n gives the degree of the homogeneous function. This result can be extended to any number of variables.

Problem

Find the degree of the homogeneous function

1. $f(x, y) = x^2 - 2xy + y^2$

2. f(x,y) = $\dfrac{x-y}{x+y}$

Euler's theorem on homogeneous function

If U= f(x,y,z) is a homogeneous function of degree n in the variables x, y & z then

$$x.\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n.u$$

Problem

Verify Euler's theorem for the following function

1. u(x,y) = $x^2 - 2xy + y^2$
2. u(x,y) = $x^3 + y^3 + z^3 - 3xyz$

INCREASING AND DECREASING FUNCTION

Increasing function

A function y= f(x) is said to be an increasing function if $f(x_1) < f(x_2)$ for all $x_1 < x_2$.

The condition for the function to be increasing is that its first order derivative is always greater than zero .

i.e $\dfrac{dy}{dx} > 0$

Decreasing function

A function y= f(x) is said to be a decreasing function if $f(x_1) > f(x_2)$ for all $x_1 < x_2$.

The condition for the function to be decreasing is that its first order derivative is always less than zero .

i.e $\dfrac{dy}{dx} < 0$

Problems

1. Show that the function y = $x^3 + x$ is increasing for all x.
2. Find for what values of x is the function y = $8 + 2x - x^2$ is increasing or decreasing ?

Maxima and Minima Function of a single variable

A function y = f(x) is said to have maximum at x = a if f(a) > f(x) in the neighborhood of the point x = a and f(a) is the maximum value of f(x) . The point x = a is also known as local maximum point.

A function y = f(x) is said to have minimum at x = a if f(a) < f(x) in the neighborhood of the point x = a and f(a) is the minimum value of f(x) . The point x = a is also known as local minimum point.

The points at which the function attains maximum or minimum are called the turning points or stationary points

A function y=f(x) can have more than one **maximum or minimum point**. Maximum of all the maximum points is called **Global maximum** and minimum of all the minimum points is called **Global minimum**.

A point at which neither maximum nor minimum is called **Saddle point**.

[Consider a function y = f(x). If the function increases upto a particular point x = a and then decreases it is said to have a maximum at x = a. If the function decreases upto a point x = b and then increases it is said to have a minimum at a point x=b.]

The necessary and the sufficient condition for the function y=f(x) to have a maximum or minimum can be tabulated as follows

	Maximum	**Minimum**
First order or necessary condition	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 0$
Second order or sufficient condition	$\frac{d^2y}{dx^2} < 0$	$\frac{d^2y}{dx^2} > 0$

Working Procedure

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

2. Equate $\frac{dy}{dx}$ =0 and solve for x. this will give the turning points of the function.

3. Consider a turning point x = a then substitute this value of x in $\frac{d^2y}{dx^2}$ and find the nature of the second derivative. If $\left(\frac{d^2y}{dx^2}\right)_{at\,x=a}$ *< 0*, then the function has a maximum value at the point x = a. If $\left(\frac{d^2y}{dx^2}\right)_{at\,x=a}$ *> 0*, then the function has a minimum value at the point x = a.

4. Then substitute x = a in the function y = f(x) that will give the maximum or minimum value of the function at x = a.

Problem

Find the maximum and minimum values of the following function

1. $y = x^3 - 3x + 1$

PERMUTATION AND COMBINATION

Fundamental Counting Principle

If a first job can be done in ***m*** ways and a second job can be done in ***n*** ways then the total number of ways in which both the jobs can be done in succession is ***m x n.***

For example, consider 3 cities Coimbatore, Chennai and Hyderabad. Assume that there are 3 routes (by road) from Coimbatore to Chennai and 4 routes from Chennai to Hyderabad. Then the total number of routes from Coimbatore to Hyderabad via Chennai is 3 x 4 =12. This can be explained as follows.

For every route from Coimbatore to Chennai there are 4 routes from Chennai to Hyderabad. Since there are 3 road routes from Coimbatore to Chennai, the total number of routes **is 3 x 4 =12.**

The above principle can be extended as follows**.** If there are *n* jobs and if there are m_i ways in which the i^{th} job can be done, then the total number of ways in which all the *n* jobs can be done in succession (1^{st} job, 2^{nd} job, 3^{rd} job… n^{th} job) is given by $m_1 \times m_2 \times m_3 \ldots \times m_n$.

Permutation

Permutation means *arrangement* of things. The word *arrangement* is used, if the order of things is considered. Let us assume that there are 3 plants P_1, P_2, P_3. These 3 plants can be planted in the following 6 ways namely

P_1	P_2	P_3
P_1	P_3	P_2
P_2	P_1	P_3
P_2	P_3	P_1
P_3	P_1	P_2
P_3	P_2	P_1

Each arrangement is called a permutation. Thus there are 6 arrangements (permutations) of 3 plants taking all the 3 plants at a time. This we write as $3P_3$. Therefore $3P_3$ = 6. Suppose out of the 3 objects we choose only 2 objects and arrange them. How many arrangements are possible? For this consider 2 boxes as shown in figure.

I Box	II Box

Since we want to arrange only two objects and we have totally 3 objects, the first box can be filled by any one of the 3 objects, (i.e.) the first box can be filled in 3 ways. After filling the first box we are left with only 2 objects and the second box can be filled by any one of these two objects. Therefore from Fundamental Counting Principle the total number of ways in which both the boxes can be filled is 3 x 2 =6. This we write as $3P_2 = 6$.

In general the number of permutations of n objects taking r objects at a time is denoted by nPr. Its value is given by

$$n\,\mathrm{Pr} = n(n-1)(n-2)\ldots(n-r+1)$$

$$= \frac{n(n-1)(n-2)\ldots.(n-r+1)\times(n-r)(n-r-1)\ldots2.1}{(n-r)(n-r-1)\ldots.2.1}$$

$$i.e \quad n\,\mathrm{Pr} = \frac{n\,!}{(n-r)!}$$

Note: 1

a) $nPn = n\,!$ (b) $nP_1 = n$. (c) $nP_0 = 1$.

Examples:

1. Evaluate $8P_3$

Solution:

$$8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8\times7\times6\times5\,!}{5\,!} = 336$$

2. Evaluate $11P_2$

Solution:

$$11P_2 = \frac{11!}{(11-2)!} = \frac{11!}{9!} = \frac{11\times10\times9!}{9\,!} = 110$$

3. There are 6 varieties on brinjal, in how many ways these can be arranged in 6 plots which are in a line?

Solution

Six varieties of brinjal can be arranged in 6 plots in $6P_6$ ways.

$$6P_6 = \frac{6!}{(6-6)!} = \frac{6!}{0!} = 6! \qquad [0! = 1]$$

$= 6 \times 5 \times 4 \times 3 \times 2\times1 = 720$.

Therefore 6 varieties of brinjal can be arranged in 720 ways.

4. There are 5 varieties of roses and 2 varieties of jasmine to be arranged in a row, for a photograph. In how many ways can they be arranged, if

(i) all varieties of jasmine together

(ii) All varieties of jasmine are not together.

Solution

i) Since the 2 varieties of jasmine are inseparable, consider them as one single unit. This together with 5 varieties of roses make 6 units which can be arranged themselves in 6! ways.

In every one of these permutations, 2 varieties of jasmine can be rearranged among themselves in 2! ways.

Hence the total number of arrangements required

= 6! x 2! = 720 x 2 = 1440.

ii)The number of arrangements of all 7 varieties without any restrictions =7! = 5040

Number of arrangements in which all varieties of jasmine are together = 1440.

Therefore number of arrangements required = 5040 -1440 = 3600.

Combinatination

Combination means *selection* of things. The word *selection* is used, when the order of thing is immaterial. Let us consider 3 plant varieties V_1, V2 & V_3. In how many ways 2 varieties can be selected? The possible selections are

1) V_1 & V_2

2) V_2 & V_3

3) V_1 & V_3

Each such selection is known as a combination. There are 3 selections possible from a total of 3 objects taking 2 objects at a time and we write $3C_2 = 3$.

In general the number of selections (Combinations) from a total of *n* objects taking *r* objects at a time is denoted by *n Cr.*

Relation between nPr and nCr

We know that

$n\,Pr = nCr \times r!$

(or) $\mathrm{nCr} = \dfrac{\mathrm{nPr}}{\mathrm{r!}}$ ------------(1)

But we know $\mathrm{nPr} = \dfrac{\mathrm{n!}}{(n-r)!}$ ---------(2)

Sub (2) in (1) we get

$$nCr = \frac{n!}{(n-r)!r!}$$

Another formula for *nCr*

We know that *nPr = n. (n-1). (n-2)…(n-r+1)*

$$\therefore \quad nCr = \frac{n.(n-1).(n-2)\,....(n-r+1)}{1.2.3\ ...\ r}$$

Example

1. Find the value of $10C_3$.

Solution:

$$10C_3 = \frac{10\,(9-1)(9-2)}{1.2.3} = \frac{10.\,9.\,8}{1.\,2.3} = 120$$

Note -1

a) $nC_0 = 1$

b) $nC_1 = n$

c) $nCn = 1$

d) $nCr = nCn\text{-}r$

Examples

1**. Find the value of $20C_{18}$**

Solution

We have $20C_{18} = 20C_{20-18} = 20C_2 = \frac{20\times 19}{1\times 2} = 190$

2. How many ways can 4 prizes be given away to 3 boys, if each boy is eligible for all the prizes?

Solution

Any one prize can be given to any one of the 3 boys and hence there are 3 ways of distributing each prize.

Hence, the 4 prizes can be distributed in 3^4= 81 ways.

3. A team of 8 students goes on an excursion, in two cars, of which one can accommodate 5 and the other only 4. In how many ways can they travel?

Solution

There are 8 students and the maximum number of students can accommodate in two cars together is 9.

We may divide the 8 students as follows

Case I: 5 students in the first car and 3 in the second

Case II: 4 students in the first car and 4 in the second

In Case I: 8 students are divided into groups of 5 and 3 in $^8C_3$ ways.

Similarly, in Case II: 8 students are divided into two groups of 4 and 4 in $^8C_4$ ways.

Therefore, the total number of ways in which 8 students can travel is $^8C_3 + {^8C_4} = 56 + 70 = 126$.

4. How many words of 4 consonants and 3 vowels can be made from 12 consonants and 4 vowels, if all the letters are different?

Solution

4 consonants out of 12 can be selected in $^{12}C_4$ ways.

3 vowels can be selected in $^4C_3$ ways.

Therefore, total number of groups each containing 4 consonants and 3 vowels

$$= {^{12}C_4} * {^4C_3}$$

Each group contains 7 letters, which can be arranging in 7! ways.

Therefore required number of words = $^{12}4 * {^4C_3} * 7!$

Physical and Economic Optimum for single input

Let y = f(x) be a response function. Here x stands for the input that is kgs of fertilizer applied per hectare and y the corresponding output that is kgs of yield per hectare.

We know that the maximum is only when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

This optimum is called physical optimum. We are not considering the profit with respect to the investment, we are interested only in maximizing the profit.

Economic optimum

The optimum which takes into consideration the amount invested and returns is called the economic optimum.

$$\frac{dy}{dx} = \frac{P_x}{P_y}$$

where $P_x \rightarrow$ stands for the per unit price of input that is price of fertilizer per kgs.

$P_y \rightarrow$ stands for the per unit price of output that is price of yield per kgs.

Population Dynamics

Definition

Model

A model is defined as a physical representation of any natural phenomena

Example: 1. A miniature building model.

2. A children cycle park depicting the traffic signals

3. Display of clothes on models in a show room and so on.

Mathematical Models

A mathematical model is a representation of phenomena by means of mathematical equations. If the phenomena is growth, the corresponding model is called a growth model. Here we are going to study the following 3 models.

1. Linear model
2. Exponential model
3. Logistic model

Linear model

The general form of a linear model is y = a+bx. Here both the variables x and y are of degree 1. In a linear growth model, the dependent variable is always the total dry weight which is noted by w and the independent variable is the time denoted by t. Hence the linear growth model is given by w = a+bt.

To fit a linear model of the form y=a+bx to the given data.

Here a and b are the parameters (or) constants to be estimated. Let us consider $(x_1,y_1),(x_2, y_2)\ldots(x_n, y_n)$ be n pairs of observations. By plotting these points on an ordinary graph sheet, we get a collection of dots which is called a scatter diagram.

In a linear model, these points lie close to a straight line. Suppose y = a+bx is a linear model to be fitted to the given data, the expected values of y corresponding to $x_1, x_2\ldots x_n$ are given by $(a+bx_1)$, $(a+bx_2),\ldots(a+bx_n)$. The corresponding observed values of y are $y_1, y_2\ldots\ldots y_n$. The difference between the observed value and the expected value is called a residual. The Principles of least squares states that the constants occurring in the curve of best fit should be chosen such that the sum of the squares of the residuals must be a minimum. Using this for a linear model we get the following 2 simultaneous equations in a and b, given by

$\Sigma y = na+b\Sigma x$ --------------------------- (1)

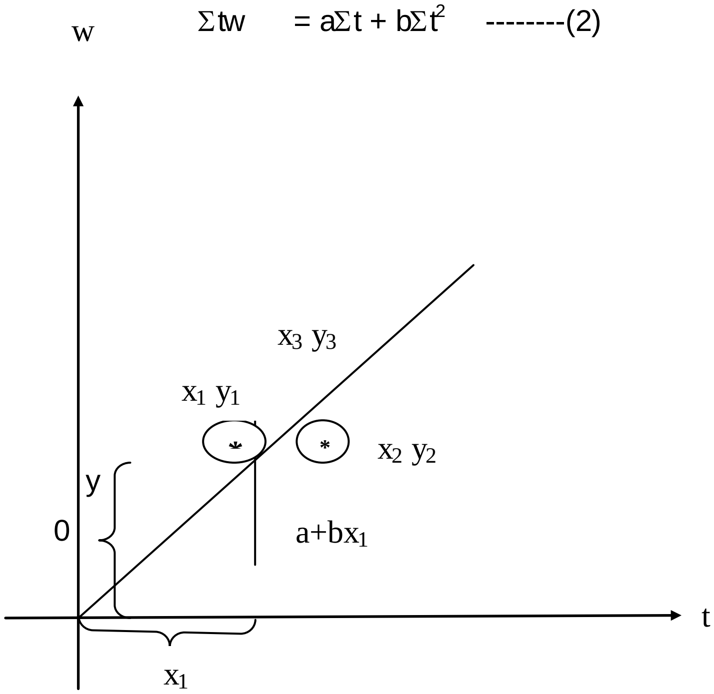
$\Sigma xy = a\Sigma x+b\Sigma x^2$ -------------------------(2)

where n is the no. of observations. Equations 1 and 2 are called **normal equations**. Given the values of x and y, we can find Σx, Σy, Σxy, Σx^2. Substituting in equations (1) and (2) we get two simultaneous equations in the constants a and b solving which we get the values of a and b.

Note: If the linear equation is w=a+bt then the corresponding normal equations become

$$\Sigma w = na + b\Sigma t \quad \text{------- (1)}$$

$$\Sigma tw = a\Sigma t + b\Sigma t^2 \quad \text{--------(2)}$$

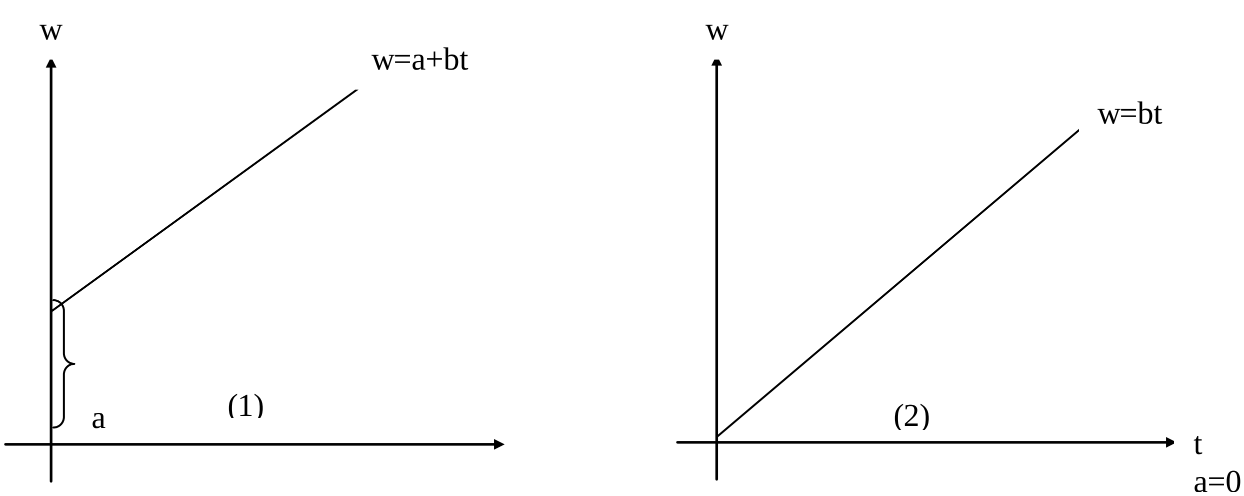


There are two types of growth models (linear)

(i) w = a+bt (with constant term)

(ii) w = bt (without constant term)

The graphs of the above models are given below :



'a' stands for the constant term which is the intercept made by the line on the w axis. When t=0, w=a ie 'a' gives the initial DMP(ie. Seed weight) ; 'b' stands for the slope of the line which gives the growth rate.

Problem

The table below gives the DMP(kgs) of a particular crop taken at different stages; fit a linear growth model of the form w=a+bt, and also calculate the estimated value of w.

t (in days) ;	0	5	10	20	25
DMP w: (kg/ha)	2	5	8	14	17

2. Exponential model

This model is of the form $w=ae^{bt}$ where a and b are constants to be determined

$$\text{Growth rate} = \frac{dw}{dt} = abe^{bt}$$

$$\text{Relative growth rate} = \frac{1}{w}.\frac{dw}{dt} = \frac{abe^{bt}}{ae^{bt}} = b$$

Here RGR = b which is also known as intensive growth rate or Malthusian parameter.

To find the parameters 'a' and 'b' in the exponential model first we convert it into a linear form by suitable transformation.

Now $w = ae^{bt}$ ----------------------(1)

Taking logarithm on both the sides we get

$\log_e w = \log_e(ae^{bt})$

$\log_e w = \log_e a + \log_e e^{bt}$

$\log w = \log_e a + bt \log_e e$

$\log w = \log_e a + bt$

$Y = A + bt$ ------------------------(2)

Where $Y = \log w$ and $A = \log_e a$

Here equation(2) is linear in the variables Y and t and hence we can find the constants 'A' and 'b' using the normal equations.

$\Sigma Y = nA + b\Sigma t$

$\Sigma tY = A\Sigma t + b\Sigma t^2$

After finding <u>A</u> by taking antilogarithms we can find the value of <u>a</u>

Note

The above model is also known as a semilog model. When the values of t and w are plotted on a semilog graph sheet we will get a straight line. On the other hand if we plot the points t and w on an ordinary graph sheet we will get an exponential curve.

Problem: Fit an exponential model to the following data.

t in days	5	15	25	35	45
W in mg per plant	0.05	0.4	2.97	21.93	162.06

3. Logistic model (or) Logistic curve

The equation of this model is given by $w = \frac{a}{1+ce^{-kt}}$ ------------- (1)

Where a, c and k are constants. The above model can be reduced to a linear form as follows:

$$1+ce^{-kt} = \frac{a}{w}$$

$$ce^{-kt} = \frac{a}{w} - 1$$

$$\frac{a}{w} - 1 = ce^{-kt}$$

Taking logarithm to the base e,

$$\log_e\left(\frac{a}{w} - 1\right) = \log c + \log e^{-kt}$$

$$\log\left(\frac{a}{w} - 1\right) = \log c - kt$$

Y = A + Bt ------------------- (2)

Where Y = $\log_e\left(\frac{a}{w} - 1\right)$

A = $\log_e c$

B = - k

Now the equation (1) is reduced to the linear form given by equation (2) using this we can determine the constants A and B from which we can get the value of the constants c and k.

Problem

The maximum dry weight of groundnut is 48 gms. The following table gives the dry matter production w of groundnut during various days estimate the logistic growth model for the following data.

t in days	25	45	60	80	105
DMP w in gm/plant	5.0	13.5	23.6	36.6	45.0

PROGRESSIONS

In this section we discuss three important series namely

1) Arithmetic Progression (A.P),

2) Geometric Progression (G.P), and

3) Harmonic Progression (H.P)

Which are very widely used in biological sciences and humanities.

Arithmetic Progressions

Consider the sequence of numbers of the form 1, 4, 7,10… . In this sequence the next term is formed by adding a constant 3 with the current term.

An arithmetic progression is a sequence in which each term (except the first term) is obtained from the previous term by adding a constant known as the common difference.

An arithmetic series is formed by the addition of the terms in an arithmetic progression.

Let the first term on an A. P. be *a* and common difference *d*.

Then, general form of an A. P is ***a*, *a* + *d*, *a* + 2*d*, *a* + 3*d*, ...**

n^{th} term of an A. P is t_n **= a + (n - 1) d**

Sum of first *n* terms of an A. P is

S_n = **n/2 [2a + (n - 1) d]**

or = **n/2 [first term + last term]**

Example 1: Find (i) The n^{th} term and (ii) Sum to *n* terms of the A.P whose first term is 2 and common difference is 3.

Answer:

1) $t_n = 2 + (n-1)3 = 3n - 1$

2) $S_n = \frac{n}{2}\left(2\times 2 + (n-1)3\right) = \frac{n}{2}(3n-1)$

Example 2: Find the sum of the first *n* natural numbers.

Solution

The sum of the natural numbers is given by

S_n=1+2+3+…+ n

This is a A.P whose first term is 1 and common difference is also one and the last term is *n.*

$$S_n = \frac{n}{2}(\text{First term} + \text{last term}) = \frac{n}{2}(n+1)$$

Example 3

Find the 15th term of the A.P 7, 17, 27,…

Solution

In the A.P 7, 17, 27,…

a =7, *d* = 17-7 =10 and *n* = 15

$t_n = a + (n-1)d$

$t_{15} = a + (15-1)d = a + 14d$

$= 7 + 14(10)$

$= 147.$

Geometric Progression

Consider the sequence of numbers

a) 1, 2, 4, 8, 16…

b) 1, $\frac{1}{4}$, $\frac{1}{16}$, $\frac{1}{64}$…

In the above sequences each term is formed by multiplying constant with the preceding term. For example, in the first sequence each term is formed by multiplying a constant 2 with the preceding term. Similarly the second sequence is formed by multiplying each term by $\frac{1}{4}$ to obtain the next term. Such a sequence of numbers is called Geometric progression (G.P).

A geometric progression is a sequence in which each term (except the first term) is derived from the preceding term by the multiplication of a non-zero constant, which is the common ratio.

The general form of G.P is a, ar, ar^2, ar^3,…

Here '*a*' is called the first term and '*r*' is called common ratio.

The n^{th} term of the G.P is denoted by t_n is given by $t_n = ar^{n-1}$

The sum of the first n terms of a G.P is given by the formula

$S_n = \frac{a(r^n - 1)}{r-1}$ if r>1

$S_n = \frac{a(1-r^n)}{1-r}$ if r<1

Examples

1. Find the common ratio of the G.P 16, 24, 36, 54.

Solution

The common ratio is $\frac{t_2}{t_1} = \frac{24}{16} = \frac{3}{2}$

2. Find the 10th term of the G.P $\frac{2}{5}$ **,** $\frac{8}{5^2}, \frac{32}{5^3}, \ldots$

Solution:

Here $a = \frac{2}{5}$ and $r = \frac{\frac{8}{5^2}}{\frac{2}{5}} = \frac{8}{25} \times \frac{5}{2} = \frac{4}{5}$

Since $t_n = ar^{n-1}$ we get

$$t_{10} = \frac{2}{5}\left(\frac{4}{5}\right)^9 = \frac{2\left(2^{18}\right)}{5^{10}} = \frac{2^{19}}{5^{10}}$$

***S*um to infinity of a G.P**

Consider the following G.P's

1). $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots.$

2). $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81} \ldots.$

In the first sequence, which is a G.P the common ratio is $r = \frac{1}{2}$.In the second G.P the common ratio is $r = -\frac{1}{3}$. In both these cases the numerical value of $r = |r|<1$.(For the first sequence $|r| = \frac{1}{2}$ and the second sequence $|r| = \frac{1}{3}$ and both are less than 1. In these equations, ie. $|r|<1$ we can find the "Sum to infinity" and it is given by the form

$$S_\infty = \frac{a}{1-r} \text{ provided } -1<r<1$$

Examples

1. Find the sum of the infinite geometric series with first term 2 and common ratio $\frac{1}{2}$**.**

Solution

Here $a = 2$ and $r = \frac{1}{2}$

$$S_\infty = \frac{2}{1-\frac{1}{2}} = 4$$

2. Find the sum of the infinite geometric series 1/2 + 1/4 + 1/8 + 1/16 + · · ·

Solution:

It is a geometric series whose first term is 1/2 and whose common ratio is 1/2, so its sum is

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=\frac{1/2}{1-\left(1/2\right)}=1$$

Harmonic Progression

Consider the sequence $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \ldots$ This sequence is formed by taking the reciprocals of the A.P *a, a+d, a+2d,…*

For example, consider the sequence $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \ldots$

Now this sequence is formed by taking the reciprocals of the terms of the A.P 2, 5, 8, 11…. Such **a sequence formed by taking the reciprocals of the terms of the A.P is called Harmonic Progression (H.P).**

The general form of the harmonic progression is $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \ldots$

The n^{th} term of the H.P is given by $t_n = \frac{1}{a+(n-1)d}$

Note

There is no formula to find the sum to *n* terms of a H.P.

Examples

1. The first and second terms of H.P are $\frac{1}{3}$ and $\frac{1}{5}$ respectively, find the 9^{th} term.

Solution

$$t_n = \frac{1}{a+(n-1)d}$$

Given *a* = 3 and *d* = 2

$$t_9 = \frac{1}{3+(9-1)2}$$

$$= \frac{1}{3+(8)2}$$

$$= \frac{1}{19}$$

Arithmetic mean, Geometric mean and Harmonic mean

The arithmetic mean (A.M) of two numbers *a & b* is defined as

$$\boxed{\text{A.M} = \frac{a+b}{2}} \qquad (1.\ 1)$$

Note: Arithmetic mean. Given *x*, *y* and *z* are consecutive terms of an A. P., then

$$y - x = z - y$$

$$2y = x + z$$

$$y = \frac{x+z}{2}$$

y is known as the arithmetic mean of the three consecutive terms of an A. P.

The Geometric mean (G.M) is defined by

$$\boxed{\text{G.M} = \sqrt{ab}} \qquad (1.\ 2)$$

The Harmonic mean (H.M) is defined as the reciprocal of the A.M of the reciprocals

ie. H.M = $\dfrac{1}{\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)}$

$$\boxed{\text{H.M} = \frac{2ab}{a+b}} \qquad (1.\ 3)$$

Examples

1 .Find the A.M, G.M and H.M of the numbers 9 & 4

Solution:

A.M = $\frac{9+4}{2} = \frac{13}{2} = 6.5$

G.M= $\sqrt{9\times 4} = \sqrt{36} = 6$

H.M= $\frac{2\times 9\times 4}{9+4} = \frac{2\times 36}{13} = 4$

2. Find the A.M,G.M and H.M between 7 and 13

Solution:

A.M = $\frac{7+13}{2} = \frac{20}{2} = 10$

G.M= $\sqrt{7\times 13} = \sqrt{91} = 9.54$

H.M= $\frac{2\times 7\times 13}{7+13} = \frac{2\times 91}{20} = 9.1$

3. If the A.M between two numbers is 1, prove that their H.M is the square of their G.M.

Solution

Arithmetic mean between two numbers is 1.

ie. $\frac{a+b}{2}$=1

$\Rightarrow\ a+b=2$

Now H.M = $\frac{2ab}{a+b} = ab$

G.M = $\sqrt{ab}$

$\therefore (\text{G.M})^2 = ab$

$\therefore \text{H.M} = (\text{G.M})^2$

Second order differential equations with constant coefficients

The general form of linear Second order differential equations with constant coefficients is

$(aD^2 + bD + c)\,y = X$ ⟶ (i)

Where a,b,c are constants and X is a function of x.and $D = \frac{d}{dx}$

When X is equal to zero, then the equation is said to be homogeneous.

Let D = m Then equation (i) becomes

$am^2 + bm + c = 0$

This is known as **auxiliary equation**. This quadratic equation has two roots say m_1 and m_2.

The solution consists of one part namely complementary function

(ie) y = complementary function

Complementary Function

Case (i)

If the roots (m_1 & m_2) are real and distinct ,then the solution is given by

$y = Ae^{m_1 x} + Be^{m_2 x}$ where A and B are the two arbitrary constants.

Case (ii)

If the roots are equal say $m_1 = m_2 = m$, then the solution is given by

$y = (A + Bx)e^{mx}$ where A and B are the two arbitrary constants.

Case (iii)

If the roots are imaginary say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$

Where α and β are real. The solution is given by $y = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$ where A and B are arbitrary constants.

Particular integral

The equation $(aD^2 + bD + c)y = X$ is called a non homogeneous second order linear equation with constant coefficients. Its solution consists of two terms complementary function and particular Integral.

(ie) y = complementary function + particular Integral

Let the given equation is f(D) y(x) = X

$$y(x) = \frac{X}{f(D)}$$

Case (i)

Let X= $e^{\alpha x}$ and f(α) $\neq 0$

Then P.I = $\frac{1}{f(D)} e^{\alpha x}$ = $\frac{1}{f(\alpha)} e^{\alpha x}$

Case (ii)

Let X = P(x) where P(x) is a polynomial

Then P.I = $\frac{1}{f(D)}$P(x) = $[f(D)]^{-1}$ P(x)

Write $[f(D)]^{-1}$ in the form $(1 \pm x)^{-1}$ $(1 \pm x)^{-2}$ and proceed to find higher order derivatives depending on the degree of the polynomial.

Newton's Law of Cooling

Rate of change in the temperature of an object is proportional to the difference between the temperature of the object and the temperature of an environment. This is known as Newton's law of cooling. Thus, if θ is the temperature of the object at time *t*, then we have

$$\frac{d\theta}{dt} \ \alpha \ \theta$$

$$\frac{d\theta}{dt} = -k(\theta)$$

This is a first order linear differential equation.

Population Growth

The differential equation describing exponential growth is

$$\frac{dG}{dt} = KG$$

This equation is called the law of growth, and the quantity K in this equation is sometimes known as the Malthusian parameter.

INVERSE OF A MATRIX

Definition

Let A be any square matrix. If there exists another square matrix B Such that AB = BA = I (I is a unit matrix) then B is called the inverse of the matrix A and is denoted by A^{-1}.

The cofactor method is used to find the inverse of a matrix. Using matrices, the solutions of simultaneous equations are found.

Working Rule to find the inverse of the matrix

Step 1: Find the determinant of the matrix.

Step 2: If the value of the determinant is non zero proceed to find the inverse of the matrix.

Step 3: Find the cofactor of each element and form the cofactor matrix.

Step 4: The transpose of the cofactor matrix is the adjoint matrix.

Step 5: The inverse of the matrix $A^{-1} = \dfrac{adj(A)}{|A|}$

Example

Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$

Solution

Let A = $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$

Step 1

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1(18-12) - 1(9-3) + (4-2)$$

$$= 6 - 6 + 2 = 2 \neq 0$$

Step 2

The value of the determinant is non zero

$\therefore A^{-1}$ exists.

Step 3

Let A_{ij} denote the cofactor of a_{ij} in $|A|$

$$A_{11} = Cofactor\ of\ 1 = (-1)^{1+1}\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} = 18 - 12 = 6$$

$$A_{12} = Cofactor\ of\ 1 = (-1)^{1+3}\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = -(9-3) = -6$$

$$A_{13} = Cofactor\ of\ 1 = (-1)^{1+3}\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 4 - 2 = 2$$

$$A_{21} = Cofactor\ of\ 1 = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -(9-4) = -5$$

$$A_{22} = Cofactor\ of\ 2 = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = 9 - 1 = 8$$

$$A_{23} = Cofactor\ of\ 3 = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -(4-1) = -3$$

$$A_{31} = Cofactor\ of\ 1 = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1$$

$$A_{32} = Cofactor\ of\ 4 = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -(3-1) = -2$$

$$A_{33} = Cofactor\ of\ 9 = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1$$

Step 4

The matrix formed by cofactors of element of determinant $|A|$ is $\begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$

$\therefore$ adj A = $\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$

Step 5

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{2}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & \frac{-5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & \frac{-3}{2} & \frac{1}{2} \end{pmatrix}$$

DEMAND FUNCTION AND SUPPLY FUNCTIONS

Demand Function

In the economics the relationship between price per unit and quantity demanded is known as demand function. Generally **when the price per unit increases, quantity demanded decreases**. Therefore if we take quantity demanded along *x* axis and the price per unit along the *y* axis then the graph will be a curve sloping downwards from left to right as shown in figure.

The demand function is generally denoted as q = f (p).

The following observations can be made form the graph.

1. The slope of the demand curve is negative.
2. Only the first quadrant part of the demand function is shown , since the price p and the quantity demanded q are positive.

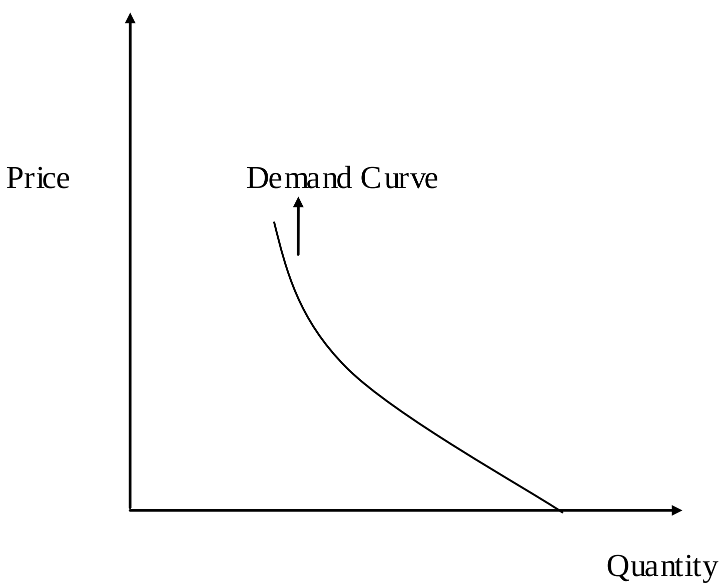


Figure : Demand

Supply Function

In economics the relationship between price per unit and the quantity supplied by the manufacturer is called supply function. Generally **when the price per unit increases, the quantity supplied also increases.** Therefore if we take the quantity supplied along the *x* axis and price per unit along the *y* axis then the graph will be a curve sloping upwards from left to right as shown in following figure.

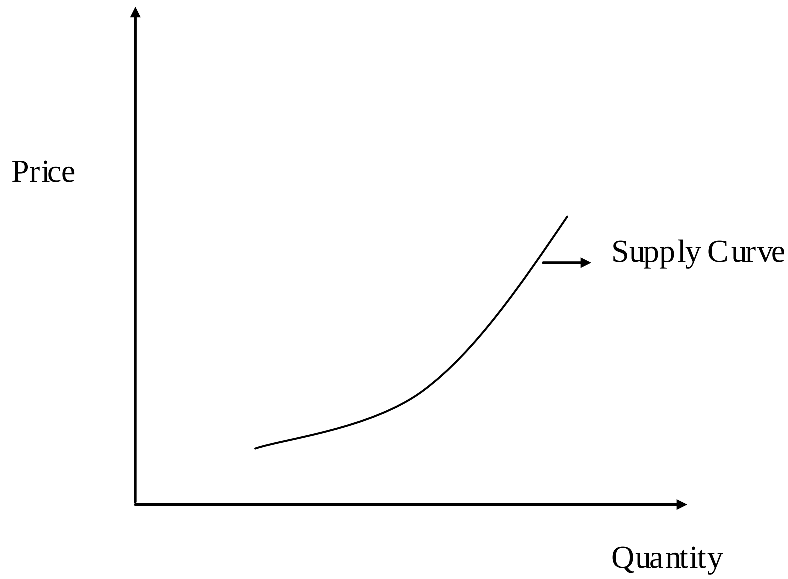


Figure: Supply Curve

Following observations can be made form supply curve.

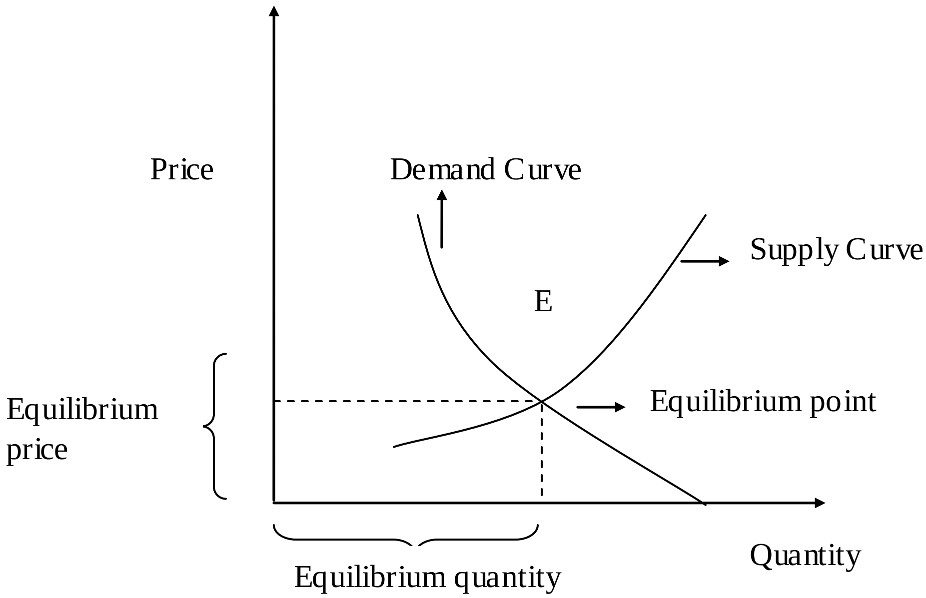
1. The slope of the supply curve is positive.
2. Only the first quadrant part of the supply function is shown, since the price p and the quantity supplied are non negative.

Equilibrium price

The price at which quantity demanded is equal to the quantity supplied is called equilibrium price.

Equilibrium quantity

The quantity obtained by substituting the equilibrium price in any one of the given demand and supply function is called equilibrium quantity. In the figure the point E is the equilibrium point in which, the x coordinate of the point E is Equilibrium quantity and the y coordinate of the point E is Equilibrium price.



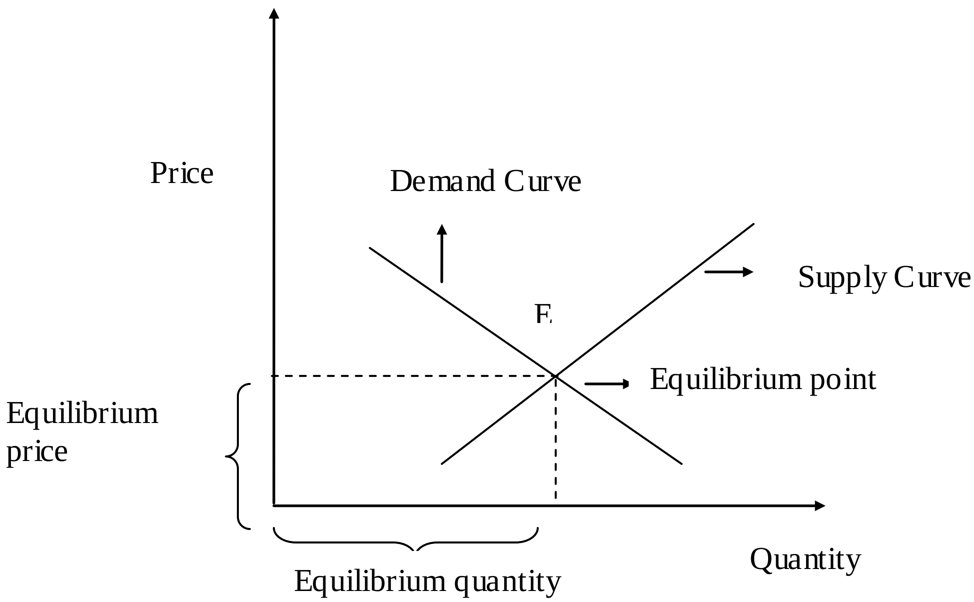
Example 1: As a simple example let us assume that both the demand and supply functions are linear. Let us assume that the demand function is given by

$$q = a + bp$$

Since the demand function slopes downwards, b is negative. Also let us assume that the supply function is given by

$$q = c + dp$$

where d is positive. The graphs of these functions are shown in the following figure



At the equilibrium point both the demand and supply are equal.

$\therefore$ $a + bp = c + dp$

i.e. $p(d-b) = a-c$

$\therefore$

$$P = \frac{a-c}{d-b}$$

This is the equilibrium price.

Example 2: Let the demand function be $q = 10 - 0.4\,p$ and supply function be $q = -5 + 0.6$ p then the equilibrium price is given by

$10 - 0.4\,p = -5 + 0.6\,p$

i.e $p = 15$

which is the equilibrium price and the equilibrium quantity is obtained by substitution this value of p either in the demand or in the supply function

$\therefore$equilibrium quantity = 10 -0.4 x 15

= 4

Examples 3: The supply and demand curves for a commodity are known to be $q_s = p-1$ and $q_d = \frac{12}{p}$ (q_s = quantity supplied; q_d = quantity demanded).Find the equilibrium price.

Solution

Equilibrium price is $q_s = q_d$

$$\therefore \quad p-1 = \frac{12}{p}$$

or, $p^2-p-12 = 0$

or, $(p+3)(p-4) = 0$

$\therefore$ $p = 4$ or -3

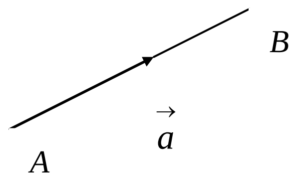
Hence, equilibrium price is 4 units.

Vector Algebra

A quantity having both magnitude and direction is called **a vector.**

Example: velocity, acceleration, momentum, force, weight etc.

Vectors are represented by directed line segments such that the length of the line segment is the magnitude of the vector and the direction of arrow marked at one end denotes the direction of the vector.



A vector denoted by $\vec{a} = \vec{AB}$ is determined by two points *A, B* such that the magnitude of the vector is the length of the line segment AB and its direction is that from *A* to *B*. The point *A* is called initial point of the vector $\vec{AB}$ and *B* is called the terminal point. Vectors are generally denoted by $\vec{a}, \vec{b}, \vec{c}$ ...(read as vector *a*, vector *b*, vector *c*,…)

Scalar

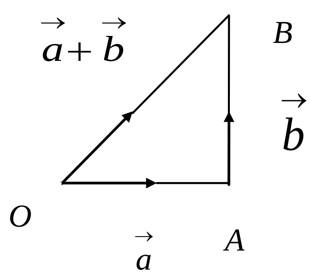
A quantity having only magnitude is called **a scalar.**

Example: mass, volume, distance etc.

Addition of vectors

If $\vec{a}$ and $\vec{b}$ are two vectors, then the addition of $\vec{a}$ from $\vec{b}$ is denoted by $\vec{a} + \vec{b}$

This is known as the triangle law of addition of vectors which states that, if two vectors are represented in magnitude and direction by the two sides of a triangle taken in the same order, then their sum is represented by the third side taken in the reverse order.



Subtraction of Vectors

If $\vec{a}$ and $\vec{b}$ are two vectors, then the subtraction of $\vec{b}$ from $\vec{a}$ is defined as the vector sum of $\vec{a}$ and -$\vec{b}$ and is denoted by $\vec{a}$ - $\vec{b}$

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

Types of Vectors

Zero or Null or a Void Vector

A vector whose initial and terminal points are coincident is called zero or null or a void vector. The zero vector is denoted by $\vec{O}$.

Proper vectors

Vectors other than the null vector are called proper vectors**.**

Unit Vector

A vector whose modulus is unity, is called **a unit vector.**

The unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$. Thus $|\hat{a}| = 1$.

There are three important unit vectors, which are commonly used, and these are the vectors in the direction of the x, y and z-axes. The unit vector in the direction of the x-axis is $\vec{i}$, the unit vector in the direction of the y-axis is $\vec{j}$ and the unit vector in the direction of the z-axis is $\vec{k}$.

Collinear or Parallel vectors

Vectors are said to be collinear or parallel if they have the same line of action or have the lines of action parallel to one another.

Coplanar vectors

Vectors are said to be coplanar if they are parallel to the same plane or they lie in the same plane.

Product of Two Vectors

There are two types of products defined between two vectors.

They are (i) Scalar product or dot product

(ii) Vector product or cross product.

Scalar Product (Dot Product)

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ is defined as the number $|\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$. It is denoted by $\vec{a}\,.\,\vec{b}$.

Properties

1. Two non-zero vectors $\vec{a}$ and $\vec{b}$ are perpendicular if $\theta = \frac{\pi}{2}$

$\therefore$ $\vec{a}\,.\,\vec{b}$ = 0

2. Let $\vec{i}, \vec{j}, \vec{k}$ be three unit vectors along three mutually perpendicular directions. Then by definition of dot product, $\vec{i}\,.\,\vec{i} = \vec{j}\,.\,\vec{j} = \vec{k}\,.\,\vec{k} = 1$ and $\vec{i}\,.\,\vec{j} = \vec{j}\,.\,\vec{k} = \vec{k}\,.\,\vec{i} = 0$

3. If m is any scalar, $\left(m\vec{a}\right).\vec{b} = \vec{a}.\left(m\vec{b}\right) = m.\left(\vec{a}.\vec{b}\right)$

4. **Scalar product of two vectors in terms of components**

   Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$: $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$.

   Then $\vec{a}\,.\,\vec{b} = \left(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\right).\left(b_1\vec{i} + b_2\vec{j} + b_3\vec{k}\right)$

   $= a_1b_1\,\vec{i}\,.\,\vec{i} + a_1b_2\,\vec{i}\,.\,\vec{j} + a_1b_3\,\vec{i}\,.\,\vec{k} + a_2b_1\,\vec{j}\,.\,\vec{i} + a_2b_2\,\vec{j}\,.\,\vec{j} + a_2b_3\,\vec{j}\,.\,\vec{k} +$

   $a_3b_1\,\vec{k}\,.\,\vec{i} + a_3b_2\,\vec{k}\,.\,\vec{j} + a_3b_3\,\vec{k}\,.\,\vec{k}$

   $= a_1b_1 + a_2b_2 + a_3b_3 \qquad \because \left[\begin{array}{l} \vec{i}\,.\,\vec{i} = \vec{j}\,.\,\vec{j} = \vec{k}\,.\,\vec{k} = 1 \text{ and} \\ \vec{i}\,.\,\vec{j} = \vec{j}\,.\,\vec{k} = \vec{k}\,.\,\vec{i} = 0 \end{array}\right]$

5. **Angle between the two vectors** $\vec{a}$ and $\vec{b}$

$$\vec{a}\,.\,\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

Work done by a force:

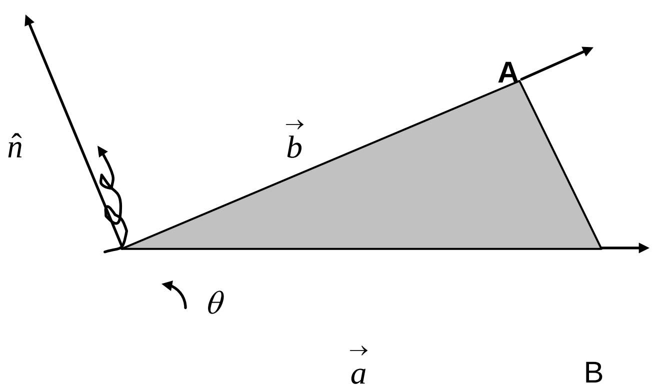
Work is measured as the product of the force and the displacement of its point of application in the direction of the force.

Let $\vec{F}$ represent a force and $\vec{d}$ the displacement of its point of application and θ is angle between $\vec{F}$ and $\vec{d}$.

$$\vec{F}\cdot\vec{d} = |\vec{F}|\,|\vec{d}|\cos\theta$$

Vector Product (Cross Product)

The vector product of two vectors $\vec{a}$ and $\vec{b}$ is defined as a vector $|\vec{a}|\,|\vec{b}|\,\sin\,\theta\hat{n}$, where θ is the angle from $\vec{a}\,to\,\vec{b}$ and $0 \le \theta \le \pi$, $\hat{n}$ is the unit vector perpendicular to $\vec{a}\,and\,\vec{b}$ such that $\vec{a}, \vec{b}, \hat{n}$ form a right handed system. It is denoted by $\vec{a}\times\vec{b}$. (Read: $\vec{a}\,cross\,\vec{b}$)



Properties

1. Vector product is not commutative

$$\vec{b}\times\vec{a} = |\vec{b}|\,|\vec{a}|\,\sin(2\pi - \theta)\,\hat{n}$$

$$= -|\vec{a}|\,|\vec{b}|\,\sin\theta\,\hat{n} \qquad [\because \sin(2\pi-\theta) = -\sin\theta]$$

$$\vec{b}\times\vec{a} = -\vec{a}\times\vec{b}$$

$$\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$$

2. Unit vector perpendicular to $\vec{a}$ *and* $\vec{b}$

$$\vec{a}\times\vec{b}=|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n} \qquad \text{.........(i)}$$

$$|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta \qquad (\because |\hat{n}|=1) \qquad \text{.........(ii)}$$

(i) ÷ (ii) gives $\hat{n} = \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

3. If two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear then $\theta = 0°$ *or* $180°$.

$$\therefore \quad \vec{a}\times\vec{b}=|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n} = |\vec{a}||\vec{b}|\,(0)\,\hat{n}=\vec{0}$$

Note

If $\vec{a}\times\vec{b} = \vec{0}$ then (i) $\vec{a} = \vec{0}$, $\vec{b}$ is any non-zero vector or

(ii) $\vec{b} = \vec{0}$, $\vec{a}$ is any non-zero or

(iii) $\vec{a}$ and $\vec{b}$ are collinear or parallel.

4. Let $\vec{i}, \vec{j}, \vec{k}$ be three unit vectors, along three mutually perpendicular directions. Then by definition of vector product $\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = \vec{0}$ *and*

$$\vec{i}\times\vec{j} = \vec{k}$$
$$\vec{j}\times\vec{i} = -k$$
$$\vec{j}\times\vec{k} = \vec{i}$$
$$\vec{k}\times\vec{j} = -i$$
$$\vec{k}\times\vec{i} = \vec{j}$$
$$\vec{i}\times\vec{k} = -\vec{j}$$

5. (m $\vec{a}$) x $\vec{b}$ = $\vec{a}$ x (m $\vec{b}$) = m($\vec{a}$ x $\vec{b}$)where m is any scalar.

6. Geometrical Meaning of the vector product of the two vectors is the area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$

Note

Area of triangle with adjacent sides $\vec{a}$ and $\vec{b} = \frac{1}{2}(\ \vec{a} \times \vec{b}\ )$

7. Vector product $\vec{a} \times \vec{b}$ in the form of a determinant

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}, \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

Then $\vec{a} \times \vec{b} = (\ a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\ ) \times (\ b_1\vec{i} + b_2\vec{j} + b_3\vec{k}\ )$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

8. The angle between the vectors $\vec{a}$ and $\vec{b}$

$$\left|\vec{a} \times \vec{b}\right| = |\vec{a}||\vec{b}|\sin\theta$$

$$\sin\theta = \frac{\left|\vec{a} \times \vec{b}\right|}{\left||\vec{a}||\vec{b}|\right|} \Rightarrow \theta = \sin^{-1}\left(\frac{\left|\vec{a} \times \vec{b}\right|}{\left||\vec{a}||\vec{b}|\right|}\right)$$

Moment of Force about a point

The moment of a force is the vector product of the displacement $\vec{r}$ and the force $\vec{F}$

(i.e) Moment $\vec{M} = \vec{r} \times \vec{F}$